Limit behavior of singular systems and the limits of value functions in optimal control

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École Doctorale de Sciences Mathématiques de Paris Centre

Thèse de doctorat

Discipline : Mathématiques Apliquées

présentée par

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Comportement limite des systèmes singuliers et les limites de fonctions valeur en contrôle optimal

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dirigée par Hélène Frankowska et Marc Quincampoix

Soutenue le 05 décembre 2014 devant le jury composé de :

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à mes parents...

Remerciements

Je voudrais remercier tout d'abord mes directeurs de thèse Hélène Frankowska et Marc Quincampoix. Je les remercie sincèrement pour la grande qualité de l'encadrement et leurs qualités scientifiques et humaines. Je leur suis très reconnaissant pour leurs conseils, contributions et le temps qu'ils ont consacré à la lecture attentive des articles presentés dans cette thèse, pour m'avoir guidé dans mon travail et m'avoir aidé à avancer.

Je voudrais également remercier les membres de jury de me faire l'honneur de leur présence.

Je veux exprimer ma profonde gratitude à tous les membres du Projet Combinatoire et Optimisation de l'Institut de Mathématiques de Jussieu et tous mes collègues du bureau des doctorants durant cette période : merci Joon, Pablo, Xiaoxi, Daniel, Marco, Teresa, Miquel, Luong, Mario et Cheng.

Je voudrais également remercier tous les responsables et membres du projet SADCO. SADCO m'a donné une expérience unique et notre réseau m'a permis de me faire de nombreux amis et collègues, avec lesquels j'espère maintenir le contact et collaborer également en avenir. Je suis très heureux d'avoir eu l'occasion d'être présent et de partager l'expérience de nombreuses conférences organisées par le réseau SADCO.

Je voudrais remercier Pablo pour son amitié pendent ces 5 dernières années et parce qu'il a toujours été là pour moi.

Merci à Jean-Baptiste pour son amitié et son hospitalité. Merci pour les conseils, pour le soutien et pour les bons moments passées ensemble.

J'exprime toute ma gratitude à Hayk Mikayelyan qui m'a beaucoup encouragé avec ses conseils pendant mes études en Autriche, Allemagne et France.

J'exprime toute ma gratitude à Theodora pour ses conseils, pour son soutien, pour l'encouragement et pour sa façon de penser qu'elle a partagé avec moi.

Je veux remercier mes très chers parents : mon père Nairi Sedrakyan, ma mère Margarit Karapetyan et toute ma famille : ma soeur Ani Sedrakyan, ma tante Varduhi Karapetyan et mon oncle Suren Karapetyan. Je tiens à remercier mon cousin Roman, mon oncle Ruben et Sassoun et leurs familles pour leur soutien pendent mes études. Je voudrais également remercier mes amis d'Arménie, en particulier : Gor, Hakob, Avet et Nina. Merci pour votre amour et votre soutien malgré la distance.

Enfin, je tiens à remercier M. Philippe Sukiasyan (le directeur de la maison des étudiants Arméniens) pour mon séjour. M. Garbis Jaloyan et Jacques pour leurs conseils et pour avoir rendu mon séjour à Paris plus confortable. Je voudrais également remercier mes amis et mes voisins, qui ont pris soin de moi lorsque j'étais malade.

Merci à tous pour tout.

Résumé

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Cette thèse se compose de deux parties principales. Dans la première partie, le Chapitre 3 est consacré à l'étude du comportement limite d'un système contrôlé singulièrement perturbé avec deux variables d'état qui sont faiblement couplées. Afin de prouver notre résultat d'approximation, nous utilisons la méthode de moyennisation et un résultat récent sur le contrôle nonexpansif. La principale nouveauté de notre approche est de permettre la dynamique limite de dépendre de l'état initial du système rapide. Notons que dans la littérature, le comportement limite d'un tel système a été généralement traité dans des conditions qui garantissent que la limite est indépendante de l'état initial du système rapide. Dans le Chapitre 4, nous généralisons les résultats du Chapitre 3 supposant une condition de nonexpansivité plus générale. De plus, nous considérons un exemple où la nouvelle condition de nonexpansivité est satisfaite, mais pas la condition de nonexpansivité du Chapitre 3. Dans la deuxième partie de la thèse, le Chapitre 5 porte sur les représentations stables des Hamiltoniens convexes associant à un Hamiltonien donné des fonctions correspondant au problème de Bolza en contrôle optimal. Dans le Chapitre 6 nous étudions également la stabilité des solutions des équations d'Hamilton-Jacobi-Bellman sous contraintes d'état en exploitant la stabilité des fonctions valeur d'une famille de problèmes de contrôle optimal de Bolza sous contraintes d'état. Nous montrons que sous des hypothèses appropriées, la fonction valeur est la solution unique d'équation d'Hamilton-Jacobi-Bellman et que les solutions sont stables par rapport à l'Hamiltonien et les contraintes d'état.

Mots-clefs

Méthode de moyennisation, perturbations singulières, condition de nonexpansivité, inclusions différentielles, équations d'Hamilton-Jacobi, contrôle optimal, représentation d'Hamiltonien, sensibilité, problème de Bolza, solution de viscosité, contraintes d'état, stabilité des solutions.

Abstract

Abstract

This thesis consists of two main parts. In the first part, Chapter 3 is devoted to the investigation of the limit behavior of a singularly perturbed control system with two state variables which are weakly coupled. In order to prove our approximation result we use the so called averaging method and a recent result on nonexpansive control. The main novelty of our averaging approach lies in the fact that the limit dynamic may depend on the initial condition of the fast system. In the literature, the investigation of the limit behavior of such systems has been usually addressed under conditions that ensure that the limit dynamic is independent from the initial condition of the fast system. In Chapter 4, we generalise the results of Chapter 3 by considering a more general nonexpansivity condition. Moreover, we consider an example where the new nonexpansity condition is satisfied but the nonexpansivity condition of Chapter 3 does not hold true. The second part deals with Hamilton-Jacobi equations under state constraints. Chapter 5 focuses on the stable representation of convex Hamiltonians by functions describing a Bolza optimal control problem. In Chapter 6 we investigate stability of solutions of Hamilton-Jacobi-Bellman equations under state constraints by studying stability of value functions of a suitable family of Bolza optimal control problems under state constraints. We show that under suitable assumptions, the value function is a unique viscosity solution to Hamilton-Jacobi-Bellman equation and that solutions are stable with respect to Hamiltonians and state constraints.

Keywords

Averaging method, singular perturbations, nonexpansivity condition, differential inclusions, Hamilton-Jacobi equations, optimal control, representation of Hamiltonians, sensitivity, Bolza problem, viscosity solution, state constraints, stability of solutions.

Table des matières

Chapitre 1

Introduction (version française)

Cette thèse se compose de deux parties principales. Dans la première partie, le Chapitre 3 est consacré à l'étude du comportement au limite d'un système contrôlé singulièrement perturbé avec deux variables d'état qui sont faiblement couplées. Dans le Chapitre 4, nous généralisons les résultats du Chapitre 3 supposant une nouvelle condition de nonexpansivité. De plus, nous considérons un exemple où la nouvelle condition de nonexpansivité est satisfaite, mais pas la condition de nonexpansivité du Chapitre 3. Dans la deuxième partie, dans le Chapitre 5 nous considérons l'équation d'Hamilton-Jacobi-Bellman et étudions la représentation stable des Hamiltoniens convexes en associant des fonctions correspondant à un problème de contrôle optimal de Bolza. Finalement, dans le Chapitre 6, nous étudions la stabilité des solutions des équations d'Hamilton-Jacobi-Bellman sous contraintes d'état en exploitant la stabilité des fonctions valeur d'une famille de problèmes de contrôle optimal de Bolza sous contraintes d'état.

1.1 Systèmes de contrôle singulièrement perturbés

1.1.1 Présentation du problème

La théorie des perturbations singulières est l'étude des problèmes (équations différentielles) avec un paramètre pour lequel les solutions du problème (à une valeur limite du paramètre) sont de natures différentes de la limite des solutions du problème général, c'est-à-dire la limite est singulière. En d'autres termes, le problème de perturbations singulières (contrairement aux perturbations régulières, où une limite peut être obtenue par la mise à zéro de la valeur du paramètre) est un problème avec un petit paramètre qui ne peut être approximée par la mise à zéro de la valeur du paramètre. Dans cette thèse, nous nous concentrons sur la classe suivante d'équations singulièrement perturbé : les systèmes lents/rapides, qui peuvent être traités avec des techniques de *moyennisation.*

Dans le Chapitre 3, nous considérons un problème de contrôle singulier de la forme suivante

$$\begin{cases} \dot{z_\varepsilon}(t) = f(y_\varepsilon(t), z_\varepsilon(t), u_\varepsilon(t)),\ u_\varepsilon(t) \in U \\ \varepsilon \dot{y_\varepsilon}(t) = g(y_\varepsilon(t), z_\varepsilon(t), u_\varepsilon(t)), \end{cases} \tag{1.1.1}$$

où U est un espace métrique, $f : \mathbb{R}^n \times \mathbb{R}^m \times U \to \mathbb{R}^n$ et $g : \mathbb{R}^n \times \mathbb{R}^m \times U \to \mathbb{R}^n$, $\varepsilon > 0$ est le paramètre de perturbation singulière, $t \in [0, T]$ la variable de temps, $z_\epsilon(\cdot) : [0, T] \to \mathbb{R}^m$ le mouvement lent, $y_\epsilon(\cdot) : [0, T] \to \mathbb{R}^n$ le mouvement rapide, et $u_\epsilon(\cdot)$ la fonction de contrôle à valeurs dans U.

Nous fixons des valeurs initiales

$$\begin{cases} z_\varepsilon(0) = z_0 \in \mathbb{R}^m \\ y_\varepsilon(0) = y_0 \in \mathbb{R}^n. \end{cases} \tag{1.1.2}$$

Le problème fondamental est de décrire le comportement des trajectoires lorsque le paramètre ε tend vers zéro.

1.1.2 Motivation

Les systèmes lents/rapides apparaissent dans la modélisation des processus du monde réel (des exemples typiques peuvent être trouvés dans [48]).

Dans la littérature, la théorie des perturbations singulières est divisée en une *théorie locale* et une *théorie globale*. Le concept de la théorie des perturbations singulières locale s'appuie sur la structure des solutions d'un problème de perturbations singulières à proximité d'un point. Néanmoins, cette théorie n'est pas évidente en raison des singularités. Le concept global de la théorie des perturbations singulières est basé sur la structure de solutions d'un problème de perturbations singulières dans un large domaine. En plus, dans des nombreuses applications ce concept donne des informations sur le comportement des solutions sur des intervalles de temps non bornés.

En absence de contrôles, c'est-à-dire pour des équations différentielles ordinaires singulièrement perturbées, il y a beaucoup de résultats de type Tychonov (cf [72], [26]) basés sur le concept local.

La méthode de réduction consiste à réduire l'équation différentielle ordinaire singulièrement perturbées à une équation algébrique-differentielle. Cette méthode exige certaines propriétés de stabilité sur le mouvement rapide.

La seconde approche est *la méthode de moyennisation*, où le mouvement rapide n'est pas explicitement pris en compte, mais son influence moyenne. Quelques résultats classiques (cf Anosov [3]) utilisent cette méthode qui est de nature globale, car elle nécessite des propriétés globales du mouvement rapide.

La situation est plus compliquée pour des systèmes *contrôlés* (singulièrement perturbés). L'approche de Gaitsgory [43] permet de traiter le problème de moyennisation en contrôle. Ceci a eté ensuite generalisé par Grammel qui montre que l'approche de moyennisation fonctionne sous des conditions beaucoup plus générales. L'auteur utilise la méthode de moyennisation dans le but de construire un champ limite pour définir une inclusion différentielle limite et il prouve que l'existence et la régularité de ce champ sont suffisant pour l'approximation du mouvement lent (de manière uniforme sur les intervalles de temps bornés) et pour donner un taux d'approximation explicite.

Dans la littérature, l'étude du comportement limite d'un système contrôlé singulièrement perturbé avec deux variables d'état faiblement couplés (cf [47]) a été généralement traité dans des conditions qui garantissent que la dynamique limite est indépendante de l'état initial du système rapide. La motivation et la nouveauté de notre approche de moyennisation est le fait que la dynamique limite peut dépendre de l'état initial du système rapide. Notre étude est basée sur une condition de non-expansivité sur le système rapide qui généralise la dissipativité ou la stabilité des propriétés de la dynamique rapide.

1.1.3 La méthode de réduction

La méthode de réduction consiste à réduire l'équation singulièrement perturbée à une équation algébrique-différentielle

$$\begin{cases} \dot{z}(t) = f(y(t), z(t), u(t)) \\ 0 = g(y(t), z(t), u(t)), \end{cases} \tag{1.1.3}$$

et de prouver que les solutions de (1.1.1) convergent vers la solution de (1.1.3). Cette approche a de nombreuses applications, depuis les travaux pionniers de Tikhonov [72].

Cette méthode exige certaines propriétés de stabilité sur le mouvement rapide. L'inconvénient de cette méthode est qu'elle nécessite des hypothèses importantes de stabilité pour la deuxième équation de (1.1.1) (cf [14, 53, 58, 60, 61, 73].

1.1.4 La méthode de moyennisation

Afin d'étudier le comportement limite du système, nous utilisons *la méthode de moyennisation.* On est intéressé par du comportement moyen des variables rapides, pour cela, nous introduisons une nouvelle variable de temps. Le problème de perturbations singulières (1.1.1) est caractérisé par le temps lent $t \in [0, T]$, nous allons également considérer un problème de perturbations singulières caractérisé par le temps rapide $\tau \in [0, T/\varepsilon]$, qui sont liés par $\tau = t/\varepsilon$. Les problèmes sont régis dans le temps lent par des systèmes d'équations différentielles singulières en $\varepsilon = 0$.

La méthode de moyennisation consiste à trouver un système limite pour la variable z et prouver la convergence. Expliquons maintenant cette méthode :

Pour tout $z \in \mathbb{R}^m$, nous considérons le système z-associé suivant

$$\begin{cases} \dot{y}(t) = g(y(t), z, u(t)) \\ y(0) = z_0, \end{cases} \tag{1.1.4}$$

et notons par $y^z(\cdot, y_0, u))$ sa solution.

Nous définissons la correspondance suivante

$$F(S, y_0, z) \doteq cl \bigcup_{u \in U} \{ \frac{1}{S} \int_0^S f(y^z(s, y_0, u), z, u(s))\, \mathrm{d}s \}.$$

Sous des hypothèses appropriées, on peut démontrer que $F(S, y_0, z)$ converge (quand $S \to \infty$) vers une certaine correspondance $\bar{F}(y_0, z)$. Le résultat principal de [46] montre que, si $F(S, y_0, z)$ converge (quand $S \to \infty$) uniformément en y_0, z vers $\bar{F}(z)$ qui est indépendante de y_0, alors les trajectoires de l'inclusion différentielle

$$\begin{cases} \dot{z}(t) \in \bar{F}(z(t)) \\ z(0) = z_0, \end{cases} \tag{1.1.5}$$

sont limites des solutions $z_\epsilon(\cdot)$ de (1.1.1) et qu'à l'inverse une solution $z_\epsilon(\cdot)$ à (1.1.1) peut être approchée par une solution de (1.1.5).

L'objectif principal consiste à etudier le cas où l'équation limite

$$\begin{cases} \dot{z}(t) \in \bar{F}(z(t), y_0) \\ z(0) = z_0, \end{cases} \tag{1.1.6}$$

pourrait dépendre de y_0. Pour ce faire, nous utilisons la méthode de moyennisation décrite ci-dessus et un résultat sur le contrôle nonexpansive [62].

1.1.5 Cas faiblement couplé

Nous limitons notre étude au cas faiblement couplé suivant

$$\begin{cases} \dot{z}_\varepsilon(t) = f(y_\varepsilon(t), z_\varepsilon(t), u_\varepsilon(t)),\ \ u_\varepsilon(t) \in U \\ \varepsilon \dot{y}_\varepsilon(t) = g(y_\varepsilon(t), u_\varepsilon(t)) \\ z_\varepsilon(0) = z_0 \\ y_\varepsilon(0) = y_0. \end{cases} \tag{1.1.7}$$

Nous supposons qu'il existe un ensemble compact $M \times N$ tel que pour tout $\varepsilon > 0$, l'ensemble $M \times N$ est invariant pour la dynamique (1.1.7), c'est-à-dire si $(y_0, z_0) \in M \times N$, pour tout contrôle $u_\epsilon(\cdot)$ la solution correspondant à (1.1.7) satisfait à $(y_\varepsilon(t), z_\varepsilon(t)) \in M \times N$ pour tous $t \geq 0$.

Nous supposons aussi une condition de non-expansivité sur g. Notre résultat principal affirme que les limites de trajectoires $z_\epsilon(\cdot)$ de (1.1.7) sont des solutions de (1.1.6). Mais, contrairement aux résultats de [46] et [43], en général, les trajectoires de (1.1.6) n'approchent pas les solutions $z_\epsilon(\cdot)$ de (1.1.7). Nous illustrons ce phénomène dans un exemple du Chapitre 3.

1.2 Solutions stables des équations d'Hamilton-Jacobi

1.2.1 L'équation d'Hamilton-Jacobi-Bellman et le problème de Bolza

Dans la théorie du contrôle optimal, nous rencontrons très souvent des équations aux dérivées partielles appelées équations d'Hamilton-Jacobi-Bellman. Dans la littérature, il existe plusieurs concepts de *solutions généralisées* des équations d'Hamilton-Jacobi, à savoir *des solutions de viscosité* (cf [23]), *des solutions contingents* (cf [28], [29]), *des solutions semi-continues inférieurement* (cf [13]). On peut prouver que, sous certaines hypothèses générales tous ces concepts sont équivalents et que la *fonction valeur* associée est la solution unique. Afin de traiter des solutions continues de l'équation d'Hamilton-Jacobi-Bellman, la notion de *solution de viscosité* a été introduite. L'idée principale est basée sur le remplacement du gradient par le *surdifférentiel* et le *sousdifférentiel*.

Dans le Chapitre 5, nous considérons l'équation suivante d'Hamilton-Jacobi-Bellman :

$$\begin{cases} -v_t(t,x) + H(t,x,-v_x(t,x)) = 0, \\ v(T,x) = \varphi(x), \end{cases} \tag{1.2.1}$$

où $T > 0$, $t \in [0,T]$, l'Hamiltonien $H : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ est convexe par rapport à la dernière variable et $\varphi : \mathbb{R}^n \to \mathbb{R}$.

Le problème qui nous intéresse est : est-il possible d'associer à H deux fonctions f : $[0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ et l : $[0,T] \times \mathbb{R}^n \times U \to \mathbb{R}$ qui héritent des propriétés de type Lipschitz de régularité de H et de telle sorte que

$$H(t,x,p) = \max_{u \in U}(\langle p, f(t,x,u)\rangle - l(t,x,u)), \tag{1.2.2}$$

où U est un sous-ensemble compact d'un espace de dimension finie.

Nos principaux résultats prouvent que nous pouvons représenter l'Hamiltonien H par de telles fonctions f et l. Nous associons à f et l le problème de Bolza de contrôle optimal, à savoir un problème d'optimisation continue de la forme suivante

$$minimize\{\varphi(x(T)) + \int_0^T l(t,x(t),u(t))\,\mathrm{d}t\}, \tag{1.2.3}$$

sur des fonctions absolument continue $x : [0,T] \to \mathbb{R}^n$ et des fonctions mesurable $u : [0,T] \to U$ vérifiant

$$\dot{x}(t) = f(t, x(t), u(t)) \ p.p. \tag{1.2.4}$$

et satisfaisant la condition initiale

$$x(0) = x_0,$$

où $x_0 \in \mathbb{R}^n$ est donné. La variable x représente l'état et u est un contrôle. Il est bien connu que sous des hypothèses appropriées un contrôle $u(\cdot)$ et la valeur initiale x_0 déterminent la trajectoire unique d'état $x(\cdot)$.

La fonction valeur associée au problème Bolza de contrôle optimal (1.2.3) - (1.2.4) est définie par : pour tout $t_0 \in [0,T]$ et $y_0 \in \mathbb{R}^n$.

$$V(t_0, y_0) = \inf\{\varphi(x(T)) + \int_{t_0}^{T} l(t, x(t), u(t))\, \mathrm{d}t : (x,u) \in S(t_0, y_0)\},$$

où $S(t_0, y_0)$ désigne l'ensemble de toutes les paires de trajectoire-contrôle du système contrôlé (1.2.4) satisfaisant la condition initial $x(t_0) = y_0$.

1.2.2 Definition de solution sous contraintes d'état

Il existe une riche littérature sur les équations d'Hamilton-Jacobi-Bellman sous contraintes d'état (cf. [19], [41]).

La solution de viscosité de l'équation d'Hamilton-Jacobi-Bellman sous contraintes d'état $x \in K$ (où K est un sous-ensemble donné de $\mathbb{R}^n$ non vide et fermé) est défini par

Definition 1.2.1. *Une fonction continue $W : [0,T] \times K \to \mathbb{R}$ est dite une solution de viscosité de l'équation d'Hamilton-Jacobi-Bellman* (1.2.1) *si $W(T,\cdot) = \varphi(\cdot)$ et si*

i) pour tout $(s,x) \in (0,T) \times K$ et tout $(p_s, p_x) \in \partial_- W(s,x)$,

$$-p_s + H(s, x, -p_x) \geq 0,$$

ii) pour tout $(s,x) \in (0,T) \times IntK$ et tout $(p_s, p_x) \in \partial_+ W(s,x)$,

$$-p_s + H(s, x, -p_x) \leq 0,$$

où ∂_-, ∂_+ désignent respectivement le sousdifférentiel et le surdifférential.

Il est bien connu que dans le cas où nous n'avons pas de contraintes d'état (sous des hypothèses appropriés) la fonction valeur est l'unique solution de viscosité de l'équation d'Hamilton-Jacobi-Bellman (cf [31], [18]).

1.2.3 Stabilité des solutions

Dans le Chapitre 6, nous étudions la stabilité des solutions des équations d'Hamilton-Jacobi-Bellman sous contraintes d'état par l'étude de la stabilité des fonctions valeur d'une famille appropriée de problèmes de Bolza en contrôle optimal sous contraintes d'état.

En absence de contrainte d'état il existe une riche littérature, où sous des hypothèses appropriées, il est prouvé que la fonction valeur correspondant à un problème de Bolza est l'unique solution de viscosité d'équation d'Hamilton-Jacobi-Bellman (cf [18], [31]). Plusieurs articles ont été consacrés à des équations d'Hamilton-Jacobi-Bellman sous contraintes d'état (cf [19], [41]). L'unicité de la solution de l'équation d'Hamilton-Jacobi-Bellman a été prouvé par différents auteurs sous des hypothèses qui comprennent une condition sur les contraintes d'état. Afin de prouver la stabilité nous imposons les hypothèses classiques sur l'Hamiltonien et une condition sur les contraintes d'état. La condition sur les contraintes d'état a un rôle crucial dans l'étude de l'unicité des solutions d'équation d'Hamilton-Jacobi-Bellman, car elle permet d'approcher (au sens de la convergence uniforme) des trajectoires *viable* (un couple $(x(\cdot), u(\cdot))$ est appelé *viable* ou *admissible* si les contraintes d'état sont satisfaites) par des trajectoires évoluant dans l'intérieur de l'ensemble K (contraintes d'état).

Dans le Chapitre 6 nous montrons aussi que sous des hypothèses appropriées, la fonction valeur de problème de Bolza sous contraintes d'état correspondant est l'unique solution de viscosité d'équation d'Hamilton-Jacobi-Bellman sous contraintes d'état.

De plus, nous montrons que les solutions sont stables par rapport à l'Hamiltonien et les contraintes d'état. C'est-à-dire, on est intéressé par la question suivante :

Si on considère des Hamiltoniens $H_i, i \geq 1$ satisfaisant des hypothèses appropriées, tel que H_i convergent uniformément sur des compacts vers un Hamiltonien H et on considère des ensembles de contraintes $K_i, i \geq 1$ (sous-ensembles à $\mathbb{R}^n$ non vides et fermés) satisfaisant des hypothèses appropriées et convergent vers un ensemble K, est-ce que des solutions de viscosité W_i d'équation d'Hamilton-Jacobi-Bellman associés à H_i et K_i convergent vers l'unique solution d'équation d'Hamilton-Jacobi-Bellman (1.2.1) associé à H et K ? Si c'est le cas, alors on dit que les solutions d'équation d'Hamilton-Jacobi-Bellman sont stables par rapport à l'Hamiltonien et les contraints d'état.

Dans le Chapitre 6, on donne une réponse à cette question en prouvant que les restrictions (à des ensembles appropriés) des solutions de viscosité W_i converge uniformément vers la restriction (aux mêmes ensembles) de l'unique solution de (1.2.1).

La preuve de ce résultat est obtenu grâce aux résultats récents sur la représentation stable des Hamiltoniens convexes [40] (cf Chapitre 5) par l'association à l'Hamiltonien des fonctions correspondantes à une famille de problèmes de Bolza, et en montrant que la fonction valeur du problème de Bolza correspondant est l'unique solution de viscosité d'équation d'Hamilton-Jacobi-Bellman. La dernière partie de la preuve est consacrée à l'étude de la stabilité des fonctions valeur de cette famille de problèmes de Bolza sous des contraintes d'état. On prouve (sous des hypothèses appropriées) la stabilité des fonctions valeur et obtient ainsi la stabilité des solutions d'équation d'Hamilton-Jacobi-Bellman.

Chapitre 2

Introduction (english version)

This thesis consists of two main parts. In the first part, Chapter 3 is devoted to the study of the limit behavior of a singularly perturbed control system with two state variables which are weakly coupled. In Chapter 4, we generalise the results of Chapter 3 by considering a new nonexpansivity condition with a corresponding norm. Moreover, we propose an example where the new nonexpansity condition is satisfied but the nonexpansivity condition of Chapter 3 does not hold true. In the second part, in Chapter 5 we consider a Hamilton-Jacobi-Bellman equation and investigate the stable representation of convex Hamiltonians by mappings corresponding to a Bolza optimal control problem. Finally, in Chapter 6 we investigate the stability of solutions of Hamilton-Jacobi-Bellman equations under state constraints by studying stability of value functions of a suitable family of Bolza optimal control problems under state constraints.

2.1 Singularly perturbed control systems

2.1.1 Problem statement

Singular perturbation theory is devoted to the investigation of problems (differential equations) with a parameter for which the solutions of the problem at a limiting value of the parameter are different in character from the limit of the solutions of the general problem, i.e. the limit is singular. In the other words, singular perturbation problem (in contrast to regular perturbation problems, where an approximation can be obtained by setting equal to zero the value of the small parameter) is a problem featuring a small parameter that cannot be approximated by setting equal to zero the value of the parameter. In this thesis we focus on the following class of singularly perturbed differential equations : coupled slow/fast systems, which can be treated with averaging techniques.

In Chapter 3 we consider a singular control problem of the following form :

$$\begin{cases} \dot{z}_\varepsilon(t) = f(y_\varepsilon(t), z_\varepsilon(t), u_\varepsilon(t)),\ u_\varepsilon(t) \in U \\ \varepsilon \dot{y}_\varepsilon(t) = g(y_\varepsilon(t), z_\varepsilon(t), u_\varepsilon(t)) \end{cases} \tag{2.1.1}$$

where U is a metric space, $f : \mathbb{R}^n \times \mathbb{R}^m \times U \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^m \times U \to \mathbb{R}^n$, $\varepsilon > 0$ is the small singular perturbation parameter, $t \in [0, T]$ is the time variable, $z_\varepsilon(\cdot) : [0, T] \to \mathbb{R}^m$ is the slow motion, $y_\varepsilon(\cdot) : [0, T] \to \mathbb{R}^n$ is the fast motion, $u_\varepsilon(\cdot)$ is the control function taking values in U.

We prescribe some initial values

$$\begin{cases} z_\varepsilon(0) = z_0 \in \mathbb{R}^m \\ y_\varepsilon(0) = y_0 \in \mathbb{R}^n. \end{cases} \tag{2.1.2}$$

The fundamental problem is to describe the behaviour of trajectories when the parameter ε tends to zero.

2.1.2 Motivation

Slow/fast systems appear in the modelling of real-world processes (some of the typical examples can be found in [48]).

In the literature singular perturbation theory is divided into a *local* theory and a *global* theory. The concept of the local singular perturbation theory is based on the structure of the solutions of a singular perturbation problem near a point. Nevertheless, this theory is nontrivial because of singularities. The concept of the global singular perturbation theory is based on the structure of the solutions of a singular perturbation problem in a large domain. Moreover, in many applications this concept gives information about the behavior of solutions during unbounded time intervals.

In the absence of control, i.e. for singularly perturbed ordinary differential equations, there are many results of Tychonov type (cf [72], [26]) based on the local concept. So called *reduction method* consists in reducing the singularly perturbed ordinary differential equation to an algebraic-differential equation. This method requires some stability properties of the fast motion.

The second approach is so called *averaging method*, where the fast motion is not considered explicitly but its average influence on the slow motion. Some classical results (cf Anosov [3]) are using this method and it is of a global nature, as it requires global properties of the fast motion.

The situation is more complicated for *controlled* singularly perturbed systems. Gaitsgory's approach [43] allows to study the averaging problem for control systems. This approach was later generalized by Grammel [46].

In [46] Grammel shows that the averaging approach works under much more general conditions. The author is using the averaging method in order to construct a limit set field defining a limit differential inclusion for the slow motion and proves that existence and regularity of the limit set field suffices to approximate the slow motion uniformly on bounded time invervals and to give explicit approximation rates.

In the literature, the investigation of the limit behavior of a singularly perturbed control system with weakly coupled two state variables (cf [47]) has been usually addressed under conditions that ensure that the limit dynamic is independent to the initial condition of the fast system. The motivation and novelty of our averaging approach is the fact that the limit dynamic may depend on the initial condition of the fast system. Our study is based on a suitable nonexpansivity condition on the fast system which generalizes dissipativity or stability properties of the fast dynamics.

2.1.3 Reduction method

Reduction method consists in elimination of fast variable and reducing the singularly perturbed equation to an algebraic-differential equation

$$\begin{cases} \dot z(t) = f(y(t), z(t), u(t)) \\ 0 = g(y(t), z(t), u(t)) \end{cases} \tag{2.1.3}$$

and in proving that solutions of (2.1.1) converge to solutions of (2.1.3). This approach has many applications since the pioneering work of Tichonov [72]. This method requires some stability properties on the fast motion. The disadvantage of this method is that it requires strong stability assumptions for the second equation of (2.1.1) (cf [53, 58, 60, 61, 73]. Nevertheless, there is a rich literature for the uncontrolled case using the reduction method.

2.1.4 Averaging method

The second approach consists in averaging over the fast variables, by rescaling the time variable. Besides singular perturbation problem (2.1.1) characterized by slow time $t \in [0, T]$ we will also consider a singular perturbation problem characterized by fast time $\tau \in [0, T/\varepsilon]$, which are related by $\tau = t/\varepsilon$. The problems are governed in slow time by systems of differential equations singular at $\varepsilon = 0$.

Averaging method consists in finding a limit dynamical system only for the z variable and in proving the convergence. In order to explain this method in more details for any $z \in \mathbb{R}^m$, consider the following associated $z-$system

$$\begin{cases} \dot{y}(t) = g(y(t), z, u(t)) \\ y(0) = y_0, \end{cases} \tag{2.1.4}$$

and denote its solution by $y^z(\cdot, y_0, u))$.

We define the following set-valued map

$$F(S, y_0, z) \doteq cl \bigcup_{u \in U} \{ \frac{1}{S} \int_0^S f(y^z(s, y_0, u), z, u(s)) \, \mathrm{d}s \}.$$

It is possible to prove (under suitable assumptions) that $F(S, y_0, z)$ converge (when $S \to \infty$) to some $\bar{F}(y_0, z)$. In [46] Grammel shows that if $F(S, y_0, z)$ converge (when $S \to \infty$) uniformly in y_0, z to a set $\bar{F}(z)$ which is independent of y_0, then the trajectories of the differential inclusion

$$\begin{cases} \dot{z}(t) \in \bar{F}(z(t)) \\ z(0) = z_0, \end{cases} \tag{2.1.5}$$

are limits of $z_\varepsilon(\cdot)$ solutions of (2.1.1) and that conversely any solution $z_\varepsilon(\cdot)$ to (2.1.1) can be approximated by a solution of (2.1.5).

Our main goal consists in investigating the case, where the limit equation

$$\begin{cases} \dot{z}(t) \in \bar{F}(z(t), y_0) \\ z(0) = z_0, \end{cases} \tag{2.1.6}$$

could depend on y_0. For doing this, we use the averaging method described above and a result on nonexpansive control [62].

2.1.5 Weakly coupled case

We restrict our consideration to the *weakly coupled* case

$$\begin{cases} \dot{z_\varepsilon}(t) = f(y_\varepsilon(t), z_\varepsilon(t), u_\varepsilon(t)), \; u_\varepsilon(t) \in U \\ \varepsilon \dot{y_\varepsilon}(t) = g(y_\varepsilon(t), u_\varepsilon(t)) \\ z_\varepsilon(0) = z_0 \\ y_\varepsilon(0) = y_0. \end{cases} \tag{2.1.7}$$

We suppose that there exists a compact set $M \times N$ such that for all $\varepsilon > 0$, the set $M \times N$ invariant for the dynamics of (2.1.7), that is, if $(y_0, z_0) \in M \times N$, then for every control $u_\varepsilon(\cdot)$ the corresponding solution to (2.1.7) satisfies $(y_\varepsilon(t), z_\varepsilon(t)) \in M \times N$ for all $t \geq 0$.

Moreover, we also assume a nonexpansivity condition on the map g. Our main result states that the limit trajectories $z_\varepsilon(\cdot)$ of (2.1.7) are solutions to (2.1.6).

But in contrast to results of [46] and [43], in general the trajectories of (2.1.6) do not approximate the solution $z_\varepsilon(\cdot)$ of (2.1.7), the illustration of this fact is given in Chapter 3 via an example.

2.2 Stable solutions of Hamilton-Jacobi-Bellman equations

2.2.1 Hamilton-Jacobi-Bellman equation and Bolza problem

In the *optimal control* theory very often we deal with partial differential equations called Hamilton-Jacobi-Bellman equations. In the literature there are several concepts of the *generalized* solutions of Hamilton-Jacobi equations, i.e. *viscosity solutions* (cf [23]), *contingent solutions* (cf [71], [28], [29]), *lower semicontinuous solutions* (cf [13]). It is known that under some general assumptions all these concepts are equivalent and that the associated *value function* is the unique solution. In order to deal with continuous solutions of the Hamilton-Jacobi-Bellman equation the notion of viscosity solution was introduced. The main idea is based on replacing the gradient by *superdifferential* and *subdifferential*.

In Chapter 5 we consider the following Hamilton-Jacobi-Bellman equation

$$\begin{cases} -v_t(t,x) + H(t,x,-v_x(t,x)) = 0, \\ v(T,x) = \varphi(x), \end{cases} \tag{2.2.1}$$

where $T > 0$, $t \in [0,T]$, the Hamiltonian $H : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is convex in the last variable and $\varphi : \mathbb{R}^n \to \mathbb{R}$.

The problem we are interested in is : whether we can associate to H mappings $f : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ and $l : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}$ inheriting Lipschitz type regularity properties of H and such that

$$H(t,x,p) = \max_{u \in U}(\langle p, f(t,x,u)\rangle - l(t,x,u)), \tag{2.2.2}$$

where U is a compact subset of a finite dimensional space.

Our main result states that we can represent the Hamiltonian H by such mappings f and l. We associate to f and l the Bolza optimal control problem, i.e. a continuous optimization problem of the following form

$$minimize\{\varphi(x(T)) + \int_0^T l(t,x(t),u(t))\,\mathrm{d}t\} \tag{2.2.3}$$

over absolutely continuous $x : [0,T] \to \mathbb{R}^n$ and measurable $u : [0,T] \to U$ such that

$$\dot{x}(t) = f(t,x(t),u(t)) \ a.e. \tag{2.2.4}$$

and

$$x(0) = x_0,$$

where $x_0 \in \mathbb{R}^n$ is given. In above the variable x stands for the *state* and u for the *control*. It is well known that under suitable assumptions a control $u(\cdot)$ and the initial value x_0 determine the unique state trajectory $x(\cdot)$.

The value function associated to the Bolza optimal control problem (2.2.3)-(2.2.4) is defined by : for all $t_0 \in [0,T]$ and $y_0 \in \mathbb{R}^n$,

$$V(t_0,y_0) = \inf\{\varphi(x(T)) + \int_{t_0}^{T} l(t,x(t),u(t))\,\mathrm{d}t : (x,u) \in S(t_0,y_0)\},$$

where $S(t_0,y_0)$ denotes the set of all trajectory-control pairs of the control system (2.2.4) satisfying the initial condition $x(t_0) = y_0$.

2.2.2 Definition of solution under state constraints

There is a rich literature on the Hamilton-Jacobi-Bellman equations under state constraints (cf. [19], [41]).

Viscosity solution of Hamilton-Jacobi-Bellman equation under state constraint $x \in K$ (where K is a given nonempty and closed subset of $\mathbb{R}^n$) is defined by

Definition 2.2.1. *A continuous function* $W : [0,T] \times K \to \mathbb{R}$ *is called a viscosity solution of Hamilton-Jacobi-Bellman equation* (2.2.1) *if* $W(T,\cdot) = \varphi(\cdot)$ *and*

i) for all $(s,x) \in (0,T) \times K$ *and all* $(p_s,p_x) \in \partial_- W(s,x)$

$$-p_s + H(s,x,-p_x) \geq 0.$$

ii) for all $(s,x) \in (0,T) \times IntK$ *and all* $(p_s,p_x) \in \partial_+ W(s,x)$

$$-p_s + H(s,x,-p_x) \leq 0,$$

where ∂_-, ∂_+ denote the subdifferential and supperdifferential, respectively.

It is well known that in the case when we have no state constraints (under suitable assumptions) the value function is the unique viscosity solution of Hamilton-Jacobi-Bellman equation (cf [31], [18]).

2.2.3 Stability of solutions

In Chapter 6 we investigate stability of solutions of Hamilton-Jacobi-Bellman equations under state constraints by studying stability of value functions of a suitable family of Bolza optimal control problems under state constraints. For the case with no state constraints there is large literature, where under appropriate assumptions it is proved that the value function of corresponding Bolza problem is the unique viscosity solution of Hamilton-Jacobi-Bellman equation, cf. [18], [31]. Several papers were devoted to Hamilton-Jacobi-Bellman equations under state constraints, cf. [19], [41]. The uniqueness of solution of Hamilton-Jacobi-Bellman equation was proved by different authors under the hypotheses which include the so called inward-pointing condition. In order to prove the stability we impose the classical assumptions on Hamiltonians and an inward pointing condition on state constraints. Inward pointing condition has a crucial role in study of uniqueness of solutions to Hamilton-Jacobi-Bellman equation under state constraints, as it permits to approximate (in the sense of uniform convergence) *feasible* trajectories (a tuple $(x(\cdot),u(\cdot))$ is called feasible or admissible if the state constraints are satisfied) by trajectories staying in the interior of the set K (the state constraints). Such approximations may be done via neighboring feasible trajectories theorems (NFT) (cf [17]).

In Chapter 6 we also show that under suitable assumptions the value function of the corresponding Bolza problem under state constraints is a unique viscosity solution to Hamilton-Jacobi-Bellman equation under state constraints.

Moreover, we prove that solutions are stable with respect to Hamiltonians and state constraints. That is, we are interested in the following question :

If we consider Hamiltonians $H_i, i \geq 1$ such that H_i converge to a Hamiltonian H uniformly on compacts and we consider sets of state constraints $K_i, i \geq 1$ (closed nonempty subsets of $\mathbb{R}^n$) satisfying appropriate assumptions and converging to a set K, does a sequence W_i of viscosity solutions to the Hamilton-Jacobi-Bellman equations associated with H_i and K_i converge to the unique solution of the Hamilton-Jacobi-Bellman equation (2.2.1) associated with H and K ? If this is the case, we say that solutions of Hamilton-Jacobi-Bellman equations are stable with respect to Hamiltonians and state constraints.

In Chapter 6 we give an answer to this question by proving that the restrictions (to appropriate compact sets) of viscosity solutions W_i converge uniformly to the restriction (to the same compact sets) of the unique solution of (2.2.1).

The proof of this result is based on the recent result on stable representation of convex Hamiltonians [40] (cf Chapter 5) via associating to Hamiltonians some mappings corresponding to a family of Bolza optimal control problems and then showing that the value function of the corresponding Bolza problem is a unique viscosity solution of Hamilton-Jacobi-Bellman equation. The last part of the proof is based on the investigation of the stability of value functions of a corresponding family of Bolza problems under state constraints. Under suitable assumptions we prove the stability of value functions and obtain in this way the stability of solutions to Hamilton-Jacobi-Bellman equations.

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Part I

Notations

B	Unit ball in $\mathbb{R}^n$
RB	Closed ball in $\mathbb{R}^n$ of center 0 and radius R
$B(x_0, r)$	Closed ball in $\mathbb{R}^n$ of center x_0 and radius r
$\partial\Omega$	Boundary of the set Ω
$\bar{\Omega}$	Closure of the set Ω
Ω^c	Complement of the set Ω
$co\Omega$	Convex hull of the set Ω
$\bar{co}\Omega$	Closed convex hull of the set Ω
$\|x\|$	Euclidean norm of $x \in \mathbb{R}^n$
$\nabla f(x)$	Gradient of the differentiable function f at x
f^*	The Fenchel conjugate of the map f
$Dom(f)$	Domain of the function f
$graph(f)$	Graph of the map (or set-valued map) f
$epi(f)$	Epigraph of the function f
$hyp(f)$	Hypograph of the function f
$\partial_p f(p, x)$	The derivative in p variable of the differentiable function f
$\partial_- f(x)$	Subdifferential of the function f at $x \in Dom(f)$
$\partial_+ f(x)$	Supperdifferential of the function f at $x \in Dom(f)$
$D_\downarrow f(x)(v)$	Upper directional derivative of the function f at x in the direction v
$D_\uparrow f(x)(v)$	Lower directional derivative of the function f at x in the direction v
$T_\Omega(x)$	Bouligand tangent cone to the set Ω at $x \in \Omega$
$N_\Omega(x)$	Bouligand normal cone to Ω at $x \in \Omega$
$f\mid_\Omega$	Restriction of the function f to the set Ω

Chapter 3

Averaging problem for weakly coupled nonexpansive control systems

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Accepted for publication in J. Nonlinear Analysis, TMA.

Abstract. This paper investigates the limit behavior of a singularly perturbed control system with two state variables which are weakly coupled. The main novelty of our averaging approach lies in the fact that the limit dynamic may depend the initial condition of the fast system while in the literature this problem has been usually addressed under conditions that ensure that the limit dynamic is independent to this initial condition. Our study is based on a suitable nonexpansivity condition on the fast system which generalized dissipativity or stability properties of the fast dynamics.

Keywords: Averaging Method, Singular Perturbations, Control, Differential Inclusions

AMS Classification: 93C70, 34E15, 34C29, 34A60.

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3.1 Introduction

We consider the following singular control problem with a slow and fast motion

$$\begin{cases} \dot{z}_\varepsilon(t) = f(y_\varepsilon(t), z_\varepsilon(t), u_\varepsilon(t)), \ u_\varepsilon(t) \in U \\ \varepsilon \dot{y}_\varepsilon(t) = g(y_\varepsilon(t), z_\varepsilon(t), u_\varepsilon(t)) \\ z_\varepsilon(0) = z_0 \\ y_\varepsilon(0) = y_0, \end{cases} \tag{3.1.1}$$

where U is a metric space, $t \in [0,T]$, $f : \mathbb{R}^n \times \mathbb{R}^m \times U \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^m \times U \to \mathbb{R}^n$. In the above $\varepsilon > 0$ is the small singular perturbation parameter, $t \in [0,T]$ is the time variable, $z_\varepsilon(\cdot) : [0,T] \to \mathbb{R}^m$ is the slow motion, $y_\varepsilon(\cdot) : [0,T] \to \mathbb{R}^n$ is the fast motion, $u_\varepsilon(\cdot)$ is the control function taking values in U and $z_\varepsilon(0) = z_0 \in \mathbb{R}^m$, $y_\varepsilon(0) = y_0 \in \mathbb{R}^n$ are the initial values.
We are interested by the behaviour of trajectories when the parameter ε tends to zero. There is a wide literature on this question. There are mainly two kind of approaches of the problem. The first one, the so called *reduction method* consists in reducing the singularly perturbed equation to an algebraic-differential equation.

$$\begin{cases} \dot{z}(t) = f(y(t), z(t), u(t)) \\ 0 = g(y(t), z(t), u(t)) \end{cases} \tag{3.1.2}$$

and to prove that solutions of (3.1.1) converge to solution of (3.1.2). This approach has many applications since the pioneering work of Tichonov [72]. Unfortunately this method requires strong stability assumptions for the second equation of (3.1.1) (cf also [53, 58, 60, 61, 73].
The second approach is the *averaging method* that we use in the present paper. It consists in finding a limit dynamical system only for the z variable and to prove the convergence. Let us explain this method now (cf [4, 6, 24, 43, 46, 47, 64, 68]). For doing this we recall the main result of [46] (cf also [43]).
For any $z \in \mathbb{R}^m$, we consider the following associated $z-$system

$$\begin{cases} \dot{y}(t) = g(y(t), z, u(t)) \\ y(0) = z_0, \end{cases} \tag{3.1.3}$$

(which solution is denoted by $y^z(\cdot, y_0, u)$) and define the following set-valued map

$$F(S, y_0, z) \doteq cl \bigcup_{u \in U} \{ \frac{1}{S} \int_0^S f(y^z(s, y_0, u), z, u(s))\, \mathrm{d}s \}.$$

Under suitable assumptions, it is possible to prove that $F(S, y_0, z)$ converge (when $S \to \infty$) to some $\bar{F}(y_0, z)$. The main result of [46] shows that if $F(S, y_0, z)$ converge (when $S \to \infty$) uniformly in y_0, z to a $\bar{F}(z)$ which is independent of y_0, then the trajectories of the differential inclusion

$$\begin{cases} \dot{z}(t) \in \bar{F}(z(t)) \\ z(0) = z_0, \end{cases} \tag{3.1.4}$$

are limit of $z_\varepsilon(\cdot)$ solutions of (3.1.1) and that conversely any solution $z_\varepsilon(\cdot)$ to (3.1.1) can be approximated by a solution of (3.1.4).

The main aim of our work consists in investigating a case where the limit equation

$$\begin{cases} \dot{z}(t) \in \bar{F}(z(t), y_0) \\ z(0) = z_0, \end{cases} \tag{3.1.5}$$

could depend on y_0. This requires a slightly different approach and technics than [43] and [46]. This is motivated by a recent result on nonexpansive control [62]. For doing this we restrict our consideration to the following weakly coupled case (also studied in [47])

$$\begin{cases} \dot{z_\varepsilon}(t) = f(y_\varepsilon(t), z_\varepsilon(t), u_\varepsilon(t)),\ u_\varepsilon(t) \in U \\ \varepsilon \dot{y_\varepsilon}(t) = g(y_\varepsilon(t), u_\varepsilon(t)) \\ z_\varepsilon(0) = z_0 \\ y_\varepsilon(0) = y_0. \end{cases} \tag{3.1.6}$$

We suppose throughout the paper that there exists a compact set $M \times N$ such that for all $\varepsilon > 0$, the set $M \times N$ invariant for the dynamics of (3.1.6).

Furthermore we assume a nonexpansivity condition on the map g. Our main result says that the limit trajectories $z_\varepsilon(\cdot)$ of (3.1.6) are solutions to (3.1.5). But in contrast to results of [46] and [43], in general the trajectories of (3.1.5) do not approximate the solution $z_\varepsilon(\cdot)$ of (3.1.6). We illustrate this phenomenon by discussing an example.

This paper is organized as follows: the main notations, assumptions and results are stated in Section 3.2. We motivate the averaging method for singularly perturbed control system in Section 3.3 and prove the main approximation theorem of the slow motion in Section 3.4. In Section 3.5 we discuss two examples: The first one illustrates the fact that the limit field $\bar{F}$ may depend on the initial value y_0. The second show a case where some solutions of the limit differential inclusion (3.1.5) cannot be approximated by the solutions of (3.1.1).

3.2 Main results

We are interested by the limit behaviour of the slow motion when the perturbation parameter $\varepsilon > 0$ tends to zero for singularly perturbed control system (SPCS in short) on the bounded time interval $[0, T]$. Denote by $\mathcal{U}$ the set of measurable controls from $\mathbb{R}_+$ to a given nonempty metric space U. The notation B stands for the closed unit ball in a metric space.

Let us consider the following system

$$\begin{cases} \dot{y}(t) = g(y(t), u(t)) \\ y(0) = y_0, \end{cases} \tag{3.2.1}$$

the unique solution of which is denoted by $t \mapsto y(t, y_0, u)$.

We denote by $G(y_0) = \{y(t, y_0, u), t \geq 0, u \in \mathcal{U}\}$ the reachable set, i.e. the set of states, that can be reached starting from y_0 by trajectories of (3.2.1).

Throughout the paper, we make the following assumptions.

Assumptions.

(A1) f, g are Lipschitz continuous with Lipschitz constant smaller or equal to $L > 0$.

(A2) there exists a compact set $M \times N \subset \mathbb{R}^n \times \mathbb{R}^m$ which is invariant by (3.1.6) for all $\varepsilon > 0$, namely if $(y_0, z_0) \in M \times N$, then for every control $u_\varepsilon(\cdot)$ the corresponding solution to (3.1.6) satisfies $(y_\varepsilon(t), z_\varepsilon(t)) \in M \times N$ for all $t \geq 0$.

(A3) (Nonexpansivity condition)
For any $y_1, y_2 \in M$, $z \in N$ and $u \in U$ there exists $v \in U$ and $C > 0$ such that

$$\begin{cases} \langle g(y_1, u) - g(y_2, v), y_1 - y_2 \rangle \le 0 \\ |f(y_1, z, u) - f(y_2, z, v)| \le C|y_1 - y_2|. \end{cases}$$

(A4) For all $y_1, y_2 \in M$, $z \in N$, $T > 0$, $u \in U$, the set

$$\{(g(y_1, u), g(y_2, v), 0)) \,|v \in U, |f(y_1, z, v) - f(y_2, z, u)| \le C|y_1 - y_2|\}$$

is closed and convex.

Remark 3.2.1. *Observe that from* ***(A1)*** *and* ***(A2)****, we deduce that some $P \ge 0$ exists such that for all $(y, z, u) \in M \times N \times U$ we have $|f(y, z, u)| + |g(y, u)| \le P$.*

When f does not depend on its third variable or when the nonexpansivity condition ***(A3)*** *is fulfilled with $u = v$, then the condition* ***(A4)*** *is implied by* ***(A1)****,* ***(A3)*** *and the following simpler assumption*

(A4') *For all $y \in M$, the set $g(y, U)$ is closed and convex.*

We are now in position to state our main result

Theorem 3.2.2. *If (A1)-(A4) hold true. There exist a Lipschitz set-valued map $\bar{F} : M \times N \to \mathbb{R}^m$ and a continuous function $\mu : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ with $\mu(\cdot, 0) = 0$ such that the following approximation property of the slow motion holds true.*

For any $T > 0$, $\varepsilon > 0$, $(y_0, z_0) \in M \times N$ and for any solution $(y_\varepsilon(\cdot), z_\varepsilon(\cdot))$ to (3.1.6)*, there exists a solution $\bar{z}(\cdot)$ to* (3.1.5) *which satisfies*

$$\max_{t \in [0,T]} |z_\varepsilon(t) - \bar{z}(t)| \le \mu(T, \varepsilon). \tag{3.2.2}$$

3.3 On the limit differential inclusion

The main result of this section concerns the construction of a limit differential inclusion driven by $\bar{F}$ which is given by the following

Proposition 3.3.1. *If (A1)-(A4) hold, then there exists a Lipschitz continous set-valued map $\bar{F} : \mathbb{R}^n \times \mathbb{R}^m \rightsquigarrow \mathbb{R}^m$ with convex, compact images and a nonincreasing function $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\lim_{s \to \infty} \beta(S) = 0$, such that*

$$d_H[\bar{F}(y_0, z), \bar{co}\{\frac{1}{S}\int_0^S f(y(\tau, y_0, u(\cdot)), z, u(\tau))\, \mathrm{d}\tau,\ u(\cdot) \in \mathcal{U}\}] \le \beta(S), \tag{3.3.1}$$

where the slow state $z \in N$ is fixed, $y(\tau, y_0, u(\cdot))$ denotes the solution of (3.2.1) *and $d_H[\cdot, \cdot]$ denotes the Hausdorff distance*[3]*.*

Proof. Before proving the above proposition we need to introduce some notations and definitions. Let $s > 0$ and $p \in B \subset \mathbb{R}^m$. We define the cost between times m and $m + s$,

$$\gamma_{m,s}(y_0, z, p, u(\cdot)) = \frac{1}{s}\int_m^{m+s} \langle f(y(\tau, y_0, u(\cdot)), z, u(\tau)), p\rangle\, \mathrm{d}\tau. \tag{3.3.2}$$

3. The Hausdorff distance between two sets E and F is defined by

$$d_H[E, F] = \min\{a > 0,\ E \subset F + aB \text{ and } F \subset E + aB\}.$$

Denote
$$F_{m,s}(y_0,z,p)=\inf_{u(\cdot)\in\mathcal{U}}\gamma_{m,s}(y_0,z,p,u(\cdot)). \tag{3.3.3}$$
Set
$$\gamma_s(y_0,z,p,u)\doteq\gamma_{0,s}(y_0,z,p,u)$$
and
$$F_s(y_0,z,p)=F_{0,s}(y_0,z,p).$$

We are interested in the limit behaviour of $F_s(y_0,z,p)$, when $s\to\infty$.
Let us introduce the following notations
$$F^-(y_0,z,p)\doteq\lim_{s\to\infty}\inf F_s(y_0,z,p) \tag{3.3.4}$$
$$F^+(y_0,z,p)\doteq\lim_{s\to\infty}\sup F_s(y_0,z,p) \tag{3.3.5}$$
The Proposition 3.3.1 follows from some auxilliary Lemmas and Proposition we will state now.

Lemma 3.3.2. *For every $m_0\in\mathbb{R}_+$ and every $p\in B$ we have*
$$\sup_{s>0}\inf_{m\le m_0}F_{m,s}(y_0,z,p)\ge F^+(y_0,z,p)\ge F^-(y_0,z,p)\ge\sup_{s>0}\inf_{m\ge 0}F_{m,s}(y_0,z,p).$$

Proof. The proof of this lemma is an adaptation of a result of [62] in our context. We give a detailed proof for the reader's convenience.

Let us prove that
$$\sup_{s>0}\inf_{m\le m_0}F_{m,s}(y_0,z,p)\ge F^+(y_0,z,p).$$
Suppose, by contradiction, that there exists $\varepsilon>0$ such that for any $s>0$ it holds
$$\inf_{m\le m_0}F_{m,s}(y_0,z,p)\le F^+(y_0,z,p)-\varepsilon. \tag{3.3.6}$$
Thus for any $s>0$ there exists $m\le m_0$ such that
$$F_{m,s}(y_0,z,p)\le F^+(y_0,z,p)-\frac{\varepsilon}{2}.$$
On the other hand,
$$F_{m,s}(y_0,z,p)=\inf_{u(\cdot)\in\mathcal{U}}\frac{1}{s}\int_m^{m+s}\langle f(y(\tau,y_0,u(\cdot)),z,u(\tau)),p\rangle\,\mathrm{d}\tau=$$
$$\inf_{u(\cdot)\in\mathcal{U}}\frac{1}{s}\Big(\int_0^{m_0+s}\langle f(y(\tau,y_0,u(\cdot)),z,u(\tau)),p\rangle\,\mathrm{d}\tau-\int_{m+s}^{m_0+s}\langle f(y(\tau,y_0,u(\cdot)),z,u(\tau)),p\rangle\,\mathrm{d}\tau$$
$$-\int_0^m\langle f(y(\tau,y_0,u(\cdot)),z,u(\tau)),p\rangle\,\mathrm{d}\tau\Big)\ge\frac{m_0+s}{s}F_{m_0+s}(y_0,z,p)-\frac{2m_0P}{s}.$$
Hence we obtain that
$$\frac{m_0+s}{s}F_{m_0+s}(y_0,z,p)-\frac{2m_0P}{s}\le F^+(y_0,z,p)-\frac{\varepsilon}{2}.$$
Passing to the $\limsup$ when $s\to\infty$, we get a contradiction.

To prove

$$F^-(y_0,z,p) \geq \sup_{s>0} \inf_{m\geq 0} F_{m,s}(y_0,z,p)$$

we proceed by a contradiction. Suppose by contradiction that the claim is false then there exists $\varepsilon > 0$, $s > 0$ such that

$$F^-(y_0,z,p) + \varepsilon \leq \inf_{m\geq 0} F_{m,s}(y_0,z,p) \tag{3.3.7}$$

$$F^-(y_0,z,p) + \varepsilon \leq F_{m,s}(y_0,z,p),$$

for any m.

We shall derive a contradiction by concatenating trajectories.

Take $T_1 > 0, T_1 = ls + r, l \in \mathbb{N}$, $r \in (0,s)$ for any $u \in \mathcal{U}$.

Thus

$$T_1\gamma_{T_1}(y_0,z,p,u) = s\gamma_{0,s}(y_0,z,p,u) + s\gamma_{s,s}(y_0,z,p,u) + ... + s\gamma_{(l-1)s,s}(y_0,z,p,u) +$$

$$+ r\gamma_{ls,r}(y_0,z,p,u) \geq ls(F^-(y_0,z,p)+\varepsilon).$$

Hence

$$\gamma_{T_1}(y_0,z,p,u) \geq \frac{T_1 - r}{T_1}(F^-(y_0,z,p)+\varepsilon),$$

for T_1 large enough it follows

$$F_{T_1}(y_0,z,p) \geq F^-(y_0,z,p) + \frac{\varepsilon}{2}. \tag{3.3.8}$$

We get a contradiction with (3.3.6) by taking $\liminf$ when $T_1 \to \infty$.

The proof of the lemma is complete. □

Proposition 3.3.3. *Assume that (A1)–(A4) hold true. Then, for all $y_1, y_2 \in M$, $z \in N$, $T > 0$, $u(\cdot) \in \mathcal{U}$, there exists $v(\cdot) \in \mathcal{U}$ such that*

$$|y(t,y_1,u(\cdot)) - y(t,y_2,v(\cdot))| \leq |y_1 - y_2|, \ \forall t \in [0,T], \tag{3.3.9}$$

and for almost every $t \in [0,T]$,

$$|f(y(t,y_1,u(\cdot)),z,u(t)) - f(y(t,y_2,v(\cdot)),z,v(t))| \leq C|y_1 - y_2|. \tag{3.3.10}$$

Proof. Fix $y_1, y_2 \in M$, $z \in N$, $T > 0$ and $u \in U$. We consider the map

$$\begin{aligned} \phi(t,x,y,l) \quad = \quad & \{(g(x,u(t)), g(y,v), 0)) \,|v \in U, \\ & |f(y,z,v) - f(x,z,u(t))| \leq C|x-y|\}. \end{aligned}$$

which has compact convex values and which is upper semicontinuous with respect to (x,y,l) and measurable in t with closed convex values due to assumptions (A1)–(A4). Observe that $\phi(t,x,y,l)$ does not depend on l. From the measurable Viability Theorem [32], assumption (A4) implies that the epigraph of the function $(x,u) \mapsto |x-y|$ is viable for the differential inclusion

$$(x'(t), y'(t), l'(t)) \in \phi(t,x(t),y(t),l(t)), \ for\ a.e.\ t \geq 0. \tag{3.3.11}$$

Therefore, starting from $(y_1, y_2, |y_1 - y_2|)$ and noticing that $l(t)$ is constant, there exists a solution $(x(.), y(.), l(.))$ of (3.3.11) satisfying

$$|x(t) - y(t)| \leq l(t) = |y_1 - y_2|, \ \forall t \geq 0.$$

From one hand we have clearly $x(.) = y(., y_1, u)$ and from the other hand, by Filippov's measurable selection Theorem there exists a control $v \in \mathcal{U}$ such that $y(.) = y(., y_2, v)$. So (3.3.9) holds true. Moreover, from the very definition of ϕ, we obtain (3.3.10).

The proof is complete. □

Now we will use the above result to build the map $\bar{F}$. For doing it we introduce the map

$$F(y_0, z, p) = \sup_{s>0} \inf_{m\geq 0} F_{m,s}(y_0, z, p). \tag{3.3.12}$$

the regularity of which is obtained in the following

Lemma 3.3.4. *There exists a constant $c > 0$ such that for all $(y_1, z_1), (y_2, z_2) \in M \times N$, all $p \in B$ and all $s > 0$, we have*

$$|F_s(y_1, z_1, p) - F_s(y_2, z_2, p)| \leq c(|y_1 - y_2| + |z_1 - z_2|)|p|.$$

Proof. Fix $\eta > 0$, there exists $u(\cdot) \in \mathcal{U}$ such that

$$F_s(y_2, z_1, p) \leq \frac{1}{s}\int_0^s \langle f(y(\tau, y_2, u(\cdot)), z_1, u(\tau)), p\rangle \, \mathrm{d}\tau \leq F_s(y_2, z_1, p) + \eta. \tag{3.3.13}$$

Using Proposition 3.3.3, associated with $u(\cdot)$there exists $v(\cdot) \in U$ such that (3.3.10) holds true.

Thus (3.3.13) yields

$$\begin{aligned} &F_s(y_1, z_1, p) - F_s(y_2, z_1, p) \leq \\ &\frac{1}{s}\int_0^s \langle f(y(\tau, y_1, v(\cdot)), z_1, v(\tau)), p\rangle \quad - \langle f(y(\tau, y_2, u(\cdot)), z_1, u(\tau)), p\rangle \, \mathrm{d}\tau + \eta \\ &\leq C|y_1 - y_2||p| + \eta. \end{aligned}$$

The Lipschitz continuity of $f(y, \cdot, u)$ which implies

$$|F_s(y_2, z_1, p) - F_s(y_2, z_2, p)| \leq L|z_1 - z_2||p|.$$

So interchanging the role of (y_1, z_1) and (y_2, z_2) and taking into account that η is arbitrary, our claim follows from the triangular inequality with $c = L + C$. □

Define now for each $m \geq 0$ the set

$$G^m(y_0) = \{y(s, y_0, u), s \leq m, u \in \mathcal{U}\},$$

(the set of states which can be reached from y_0 before time m), thus

$$G(y_0) = \bigcup_{m\geq 0} G^m(y_0).$$

We recall the following easy result due to the boundedness of $G(y_0)$ (cf Lemma 3.9 in [62])

Lemma 3.3.5. *For any $\varepsilon > 0$ there exists $m_0 > 0$, such that for any $x \in G(y_0)$ there exists $x_1 \in G^{m_0}(y_0)$, such that $|x - x_1| \leq \varepsilon$.*

Lemma 3.3.6. *For any $(y_0, z) \in M \times N$ and $|p| \leq 1$*

$$\lim_{s\to\infty} F_s(y_0, z, p) = F(y_0, z, p).$$

Proof. By Lemma 3.3.2 it is sufficient to prove that for any $\varepsilon > 0$ there exists m_0 such that

$$\sup_{s>0} \inf_{m\leq m_0} F_{m,s}(y_0, z, p) \leq \sup_{s>0} \inf_{m\geq 0} F_{m,s}(y_0, z, p) + 2\varepsilon. \tag{3.3.14}$$

By Lemma 3.3.5 there exists m_0 such that for any $x \in G(y_0)$ there exists $x_1 \in G^{m_0}(y_0)$, such that $|x - x_1| \leq \varepsilon$.
Notice that for any $s > 0$

$$\inf_{m\geq 0} F_{m,s}(y_0, z, p) = \inf\{F_s(x, z, p), x \in G(y_0)\}. \tag{3.3.15}$$

and

$$\inf_{m\leq m_0} F_{m,s}(y_0, z, p) = \inf\{F_s(x, z, p), x \in G^{m_0}(y_0)\}.$$

Let $x \in G(y_0)$ be such that

$$F_s(x, z, p) \leq \inf_{m\geq 0} F_{m,s}(y_0, z, p) + \varepsilon$$

and consider $x_1 \in G^{m_0}(y_0)$ such that $|x - x_1| \leq \varepsilon$.

By Lemma 3.3.4 it follows

$$|F_s(x, z, p) - F_s(x_1, z, p)| \leq c\varepsilon.$$

Hence

$$\inf_{m\leq m_0} F_{m,s}(y_0, z, p) \leq F_s(x_1, z, p) \leq F_s(x, z, p) + c\varepsilon \leq \inf_{m\geq 0} F_{m,s}(y_0, z, p) + (c + 1)\varepsilon.$$

Passing to the supremum on s ends the proof. □

As consequence of Lemma 3.3.4 and Lemma 3.3.6 we obtain

Corollary 3.3.7. *$F(y_0, z, p)$ is Lipschitz continuous with respect to y_0 and z.*

Now let us proceed the proof of Proposition 3.3.1.

End of the proof of Proposition 3.3.1.
In view of Lemma 3.3.4 and Lemma 3.3.6 we have obtained that

$$\inf_{u(\cdot)\in\mathcal{U}} \frac{1}{s} \int_0^s \langle f(y(\tau, y_0, u(\cdot)), z, u(\tau)), p\rangle \, \mathrm{d}\tau \to_{s\to\infty} F(y_0, z, p). \tag{3.3.16}$$

Let us define

$$\bar{F}(y_0, z) = \{\xi \in \mathbb{R}^m;\ \langle \xi, p\rangle \leq F(y_0, z, p),\ \forall p \in \mathbb{R}^m\}.$$

The set-valued map $\bar{F}(\cdot, \cdot)$ has clearly compact convex values which are bounded by P. We claim that $\bar{F}$ is Lipschitz continuous with Lipschitz constant less or equal to c.

Let us fix $\xi_1 \in \bar{F}(y_1, z_1)$. By Lemma 3.3.4, we know that

$$\langle \xi_1, p\rangle \leq F(y_1, z_1, p) \leq F(y_2, z_2, p) + c|p| \left(|y_1 - y_2| + |z_1 - z_2|\right).$$

Therefore, for all $p \in B$, there exists $\nu \in B$ such that

$$\langle \xi_1 - c|p| \left(|y_1 - y_2| + |z_1 - z_2|\right) \nu, p\rangle \leq F(y_2, z_2, p).$$

In particular, we deduce that

$$\max_{p\in B}\min_{u\in B}\langle \xi_1 - c|p|\,(|y_1-y_2|+|z_1-z_2|)\,\nu, p\rangle - F(y_2,z_2,p) \leq 0. \tag{3.3.17}$$

The function $(p,\nu) \to \langle \xi_1 - c|p|\,(|y_1-y_2|+|z_1-z_2|)\,u, p\rangle - F(y_2,z_2,p)$ is linear in ν, concave in p (from Lemma 3.3.6 and because $p \mapsto F_s(y,z,p)$ is concave as the infimum of linear function) and continuous. Therefore, the Von Neumann Min-Max Theorem applied in (3.3.17), gives

$$\min_{\nu\in B}\max_{p\in B}\langle \xi_1 - c|p|\,(|y_1-y_2|+|z_1-z_2|)\,\nu, p\rangle - F(y_2,z_2,p) \leq 0.$$

Thus, there exists $\bar\nu \in B$ such that, for all $p \in B$,

$$\langle \xi_1 - c|p|\,(|y_1-y_2|+|z_1-z_2|)\,\bar\nu, p\rangle - F(y_2,z_2,p) \leq 0.$$

Now, by the very definition of $\bar F(y_2,z_2)$, we have that

$$\xi_1 - c|p|\,(|y_1-y_2|+|z_1-z_2|)\,\bar\nu \in \bar F(y_2,z_2).$$

Thus, there exists $\xi_2 \in \bar F(y_2,z_2)$ such that

$$\xi_2 = \xi_1 - c|p|\,(|y_1-y_2|+|z_1-z_2|)\,\bar u \in \bar F(y_1,z_1) + c|p|(|y_1-y_2|+|z_1-z_2|)B.$$

Which ends the proof of our claim.

By Lemma 3.3.4 the family $\{F_s(\cdot,\cdot,p)\}_{p\in B}$ is equi-continuous on $M\times N$. Using Arzela-Ascoli theorem and by Lemma 3.3.6 we deduce that $F_s(y_0,z,p)$ converge uniformly to $F(y_0,z,p)$ on $M\times N$, when $s\to\infty$.

A straightforward verification shows that

$$\beta(s,p) = \sup_{(y_0,z)\in M\times N} |F_s(y_0,z,p) - F(y_0,z,p)|.$$

is homogeneous and continuous with respect to p. By replacing $\beta(s,p)$ by $\sup_{\sigma\in(0,s]}\beta(\sigma,p)$ we obtain that $\beta(s,p)$ is nonincreasing with respect to s.

Therefore by taking

$$\beta(s) = \sup_{p\in B}\beta(s,p),$$

we obtain in view of Lemma 3.3.6 that $\beta(s)\to 0$, as $s\to\infty$ and β is nonincreasing.

Now we proceed with the proof of (3.3.1).

$$d_H[\bar F(y_0,z), \bar{co}\{\frac{1}{S}\int_0^S f(y(\tau,y_0,u(\cdot)),z,u(\tau))\,\mathrm{d}\tau,\ u(\cdot)\in\mathcal{U}\}] \leq \beta(S).$$

We first claim that

$$\bar F(y_0,z) \subset \bar{co}\{\frac{1}{S}\int_0^S f(y(\tau,y_0,u(\cdot)),z,u(\tau))\,\mathrm{d}\tau,\ u(\cdot)\in\mathcal{U}\} + \beta(S)B. \tag{3.3.18}$$

By contradiction suppose that the above inclusion is false, then there exists

$$\xi \in \bar F(y_0,z) \setminus \bar{co}\{\frac{1}{S}\int_0^S f(y(\tau,y_0,u(\cdot)),z,u(\tau))\,\mathrm{d}\tau,\ u(\cdot)\in\mathcal{U}\} + \beta(S)B.$$

Then from the separation theorem there exists some p with $|p| = 1$ such that

$$\begin{aligned} <\xi, p> \quad &> \quad <\frac{1}{S}\int_0^S f(y(\tau, y_0, u(\cdot)), z, u(\tau))\,\mathrm{d}\tau, p> +\beta(S) \\ &\geq \quad F_S(y_0, z, p) + \beta(S) \\ &\geq \quad F(y_0, z, p) - \beta(S, p) + \beta(S) \\ &\geq \quad F(y_0, z, p), \end{aligned}$$

$(\beta(S, p) - \beta(S) \leq 0$, from the very definition of $\beta(S, p))$.
Thus

$$<\xi, p> \, > F(y_0, z, p),$$

which is a contradiction with the fact that $\xi \in \bar{F}(y_0, z)$.
Our first claim is proved.

Second we show that

$$\{\frac{1}{S}\int_0^S f(y(\tau, y_0, u(\cdot)), z, u(\tau))\,\mathrm{d}\tau,\ u(\cdot) \in \mathcal{U}\} \subset \bar{F}(y_0, z) + \beta(S)B. \tag{3.3.19}$$

Assume by contradiction that it is false then there would exists some $u \in \mathcal{U}$ such that

$$\{\frac{1}{S}\int_0^S f(y(\tau, y_0, u(\cdot)), z, u(\tau))\,\mathrm{d}\tau,\ u(\cdot) \in \mathcal{U}\} \notin \bar{F}(y_0, z) + \beta(S)B.$$

The right-hand side of the above inequality being closed and convex, we can deduce from the separation theorem that there exists some p with $|p| = 1$ such that

$$<\frac{1}{S}\int_0^S f(y(\tau, y_0, u(\cdot)), z, u(\tau))\,\mathrm{d}\tau, p> \, < \, <\xi, p> +\beta(S) <\nu, p>,$$

for all $\xi \in \bar{F}(y_0, z)$ and $\nu \in B$.
Taking $\nu = -p$ we obtain

$$\begin{aligned} F(y_0, z, p) - \beta(S) \leq F_S(y_0, z, p) \quad &\leq \, <\tfrac{1}{S}\textstyle\int_0^S f(y(\tau, y_0, u(\cdot)), z, u(\tau))\,\mathrm{d}\tau, p> \\ &> \, <\xi, p> -\beta(S). \end{aligned}$$

This implies

$$F(y_0, z, p) < \, <\xi, p>$$

which is a contradiction with $\xi \in \bar{F}(y_0, z)$. So our claim (3.3.19) is obtained.
From (3.3.18) and (3.3.19) we deduce (3.3.1).
The proof of Proposition 3.3.1 is complete. □

Once $\bar{F}$ is obtained, we need a result more precise than Proposition 3.3.1.

Proposition 3.3.8. *If (A1)-(A4) hold true, then for all $s > 0$ and $m \geq 1$*

$$d_H[\bar{F}(y_0, z), \bar{co}\{\frac{1}{s}\int_m^{s+m} f(y(\tau, y_0, u(\cdot)), z, u(\tau))\,\mathrm{d}\tau,\ u(\cdot) \in \mathcal{U}\}] \leq 2\beta(s) + \frac{2m}{m+s}P. \tag{3.3.20}$$

Proof. Fix $p \in B$, $y_0 \in M, z \in N$.

First observe that, as $\beta(s,p)$ is nonincreasing with respect to s we have

$$|F_{m+s}(y_0,z,p) - F(y_0,z,p)| \leq \beta(s+m,p) \leq \beta(s,p). \tag{3.3.21}$$

Now we claim that

$$|F_{m+s}(y_0,z,p) - F_{m,s}(y_0,z,p)| \leq \beta(s,p) + \frac{2m}{m+s}P|p|. \tag{3.3.22}$$

Take $u_m \in \mathcal{U}$ which is $\beta(s,p)$ optimal in the definition of $F_{m,s}(y_0,z,p)$, that is

$$F_{m,s}(y_0,z,p) \leq \gamma_{m,s}(y_0,z,p,u_m) \leq F_{m,s}(y_0,z,p) + \beta(s,p).$$

We have

$$\begin{aligned}
F_{m+s}(y_0,z,p) - F_{m,s}(y_0,z,p) &\leq \gamma_{m+s}(y_0,z,p,u_m) - \gamma_{m,s}(y_0,z,p,u_m) + \beta(s,p) = \\
= \frac{1}{m+s}\int_0^{m+s} \langle f(y(\tau,y_0,u_m),z,u_m),p\rangle \,\mathrm{d}\tau &- \frac{1}{s}\int_m^{m+s} \langle f(y(\tau,y_0,u_m),z,u_m),p\rangle \,\mathrm{d}\tau \\
+\beta(s,p) = s\Big(\frac{1}{m+s} - \frac{1}{s}\Big)\frac{1}{s}&\int_m^{m+s} \langle f(y(\tau,y_0,u_m),z,u_m),p\rangle \,\mathrm{d}\tau - \\
-\frac{m}{m+s}\frac{1}{m}&\int_0^{m} \langle f(y(\tau,y_0,u_m),z,u_m),p\rangle \,\mathrm{d}\tau + \beta(s,p) \leq \\
&\leq \frac{m}{m+s}P|p| + \frac{m}{m+s}P|p| + \beta(s,p).
\end{aligned}$$

Thus

$$F_{m+s}(y_0,z,p) - F_{m,s}(y_0,z,p) \leq \frac{2m}{m+s}P|p| + \beta(s,p). \tag{3.3.23}$$

Let us take now $v_m \in \mathcal{U}$ which is $\beta(s,p)$ optimal for $F_{m+s}(y_0,z,p)$, that is

$$\gamma_{m+s}(y_0,z,p,v_m) \leq F_{m+s}(y_0,z,p) + \beta(s,p).$$

We obtain

$$\begin{aligned}
F_{m+s}(y_0,z,p) - F_{m,s}(y_0,z,p) &\geq \gamma_{m+s}(y_0,z,p,v_m) - \gamma_{m,s}(y_0,z,p,v_m) - \\
-\beta(s,p) &= \frac{m}{m+s}\frac{1}{s}\int_m^{m+s} \langle f(y(\tau,y_0,v_m),z,v_m),p\rangle \,\mathrm{d}\tau - \\
-\frac{m}{m+s}\frac{1}{m}\int_0^{m} \langle f(y(\tau,y_0,v_m),z,v_m),p\rangle \,\mathrm{d}\tau &- \beta(s,p) \geq -\frac{2m}{m+s}P|p| - \beta(s,p).
\end{aligned}$$

Therefore

$$F_{m+s}(y_0,z,p) - F_{m,s}(y_0,z,p) \geq -\frac{2m}{m+s}P|p| - \beta(s,p). \tag{3.3.24}$$

Inequalities (3.3.24) and (3.3.23) yield our claim (3.3.22).

From (3.3.21) and (3.3.22), using the triangular inequality we have

$$|F_{m,s}(y_0,z,p) - F(y_0,z,p)| \leq 2\beta(s,p) + \frac{2m}{m+s}P|p|.$$

Using the above relation and arguing exactly as in the proof of Proposition 3.3.1, we deduce the inequality (3.3.20).

Which ends the proof.

□

We denote

$$\tilde{\beta}(m,s) \doteq 2\beta(s) + \frac{2m}{m+s}P, \tag{3.3.25}$$

which is increasing in m.

Consequently we have that for any $m \geq 1$, $s > 0$, $y_0 \in M$, $z \in N$

$$d_H[\bar{F}(y_0,z), \bar{co}\{\frac{1}{s}\int_m^{s+m} f(y(\tau,y_0,u(\cdot)),z,u(\tau))\,\mathrm{d}\tau,\ u(\cdot)\in\mathcal{U}\}] \leq \tilde{\beta}(m,s).$$

3.4 Convergence of trajectories: Proof of main result

In this section we give the proof of Theorem 3.2.2. We fix $\varepsilon > 0$. We will build μ and $\bar{z}(\cdot)$ which satisfy (3.2.2).

Let us first substitute $t \doteq \varepsilon\tau$, i.e.

$$\begin{cases} \dot{z_\varepsilon}(\tau) = \varepsilon f(y_\varepsilon(\tau), z_\varepsilon(\tau), u_\varepsilon(\tau)) \\ \dot{y_\varepsilon}(\tau) = g(y_\varepsilon(\tau), u_\varepsilon(\tau)) \\ z_\varepsilon(0) = z_0 \\ y_\varepsilon(0) = y_0, \end{cases} \tag{3.4.1}$$

where $\tau \in [0, T/\varepsilon]$.

Let us divide the time interval $[0,T/\varepsilon]$ in the following $[\tau_\varepsilon^i, \tau_\varepsilon^{i+1}]$, $i \in I_\varepsilon \doteq (0, ..., [T/S_\varepsilon \varepsilon])$ subintervals with the same length $S_\varepsilon > 0$ (except the last one), such that $\varepsilon \mapsto S_\varepsilon$ is continuous

$$\lim_{\varepsilon\to 0} S_\varepsilon = \infty$$

and

$$\lim_{\varepsilon\to 0} \varepsilon S_\varepsilon = 0.$$

Let us denote by $t_\varepsilon^i = \varepsilon\tau_\varepsilon^i$ the corresponding sequence in slow time scale.

Consider the approximating system (3.4.2) with the z variable is constant and equal to $z_\varepsilon(\tau_\varepsilon^i)$ for $\tau \in [\tau_\varepsilon^i, \tau_\varepsilon^{i+1})$.

$$\begin{cases} \dot{\bar{z}}_\varepsilon(\tau) = \varepsilon f(y_\varepsilon(\tau), z_\varepsilon(\tau_\varepsilon^i), u_\varepsilon(\tau)) \\ \dot{y_\varepsilon}(\tau) = g(y_\varepsilon(\tau), u_\varepsilon(\tau)) \\ \bar{z_\varepsilon}(0) = z_0 \\ \bar{y_\varepsilon}(0) = y_0 \end{cases} \tag{3.4.2}$$

We denote by $\bar{z}_\varepsilon(\cdot)$ the solution to (3.4.2).

By Proposition 3.3.8, there exists $\alpha_i^\varepsilon \in \bar{F}(y_0, z_\varepsilon(\tau_\varepsilon^i))$ such that

$$|\alpha_i^\varepsilon - \frac{1}{S_\varepsilon}\int_{\tau_\varepsilon^{i-1}}^{\tau_\varepsilon^i} f(y_\varepsilon(\tau,y_0,u_\varepsilon(\cdot)), z_\varepsilon(\tau_\varepsilon^i), u_\varepsilon(\tau))\,\mathrm{d}\tau| \leq \tilde{\beta}(\tau_\varepsilon^{i-1}, S_\varepsilon) \leq \tilde{\beta}(\frac{T}{\varepsilon}, S_\varepsilon), \tag{3.4.3}$$

because the function $\tilde{\beta}$ is increasing with respect to the first variable.

Now let us define

$$\eta_0 = z_0$$

and

$$\eta_{i+1} = \eta_i + \varepsilon S_\varepsilon \alpha_i^\varepsilon,$$

for $i \in I_\varepsilon$ and

$$\eta(t) = \eta_i + \alpha_i^\varepsilon (t - t_\varepsilon^i), \tag{3.4.4}$$

for any $t \in [t_\varepsilon^i, t_\varepsilon^{i+1}]$.

Thus from (3.4.2) and (3.4.3), we have

$$\begin{aligned} |\bar{z}_\varepsilon(\tau_\varepsilon^i) - \eta_i| = |\bar{z}_\varepsilon(\tau_\varepsilon^{i-1}) + \varepsilon \int_{\tau_\varepsilon^{i-1}}^{\tau_\varepsilon^i} f(y_\varepsilon(\tau), z_\varepsilon(\tau_\varepsilon^i), u_\varepsilon(\tau))\, \mathrm{d}\tau - \eta_{i-1} - \varepsilon S_\varepsilon \alpha_i^\varepsilon| \\ \leq |\bar{z}_\varepsilon(\tau_\varepsilon^{i-1}) - \eta_{i-1}| + \varepsilon S_\varepsilon \tilde{\beta}(\frac{T}{\varepsilon}, S_\varepsilon). \end{aligned}$$

By iteration we deduce that

$$|\bar{z}_\varepsilon(\tau_\varepsilon^i) - \eta_i| \leq \tilde{\beta}(\frac{T}{\varepsilon}, S_\varepsilon)\varepsilon S_\varepsilon i \leq T\tilde{\beta}(\frac{T}{\varepsilon}, S_\varepsilon) \text{ for } i \in I_\varepsilon. \tag{3.4.5}$$

Using the Lipschitz continuity of $\bar{F}(\cdot, \cdot)$ we obtain for $t \in [t_\varepsilon^i, t_\varepsilon^{i+1}]$

$$\begin{aligned} d(\dot{\eta}(t), \bar{F}(y_0, \eta(t))) = d(\alpha_i^\varepsilon, \bar{F}(y_0, \eta(t))) \\ \leq dist(\alpha_i^\varepsilon, \bar{F}(y_0, \bar{z}_\varepsilon(\tau_\varepsilon^i)) + c(|\eta(t) - \eta_i| + |\eta_i - \bar{z}_\varepsilon(\tau_\varepsilon^i)|) \\ \leq cT\tilde{\beta}(\frac{T}{\varepsilon}, S_\varepsilon) + cP\varepsilon S_\varepsilon. \end{aligned}$$

By Filippov theorem [4] there exists $\bar{z}(\cdot)$ solution of

$$\begin{cases} \dot{\bar{z}}(\tau) \in \bar{F}(y_0, \bar{z}(\tau)) \\ \bar{z}(0) = z_0, \end{cases}$$

such that

$$|\eta(t) - \bar{z}(t)| \leq cTe^{cT}(T\tilde{\beta}(\frac{T}{\varepsilon}, S_\varepsilon) + cP\varepsilon S_\varepsilon). \tag{3.4.6}$$

Now we proceed by estimating $|\bar{z}(t) - z_\varepsilon(t)|$ in order to obtain our main equation (3.2.2).

For doing this, we pass to the fast variable τ. By (3.4.6), (3.4.5), (3.4.1) and (3.4.2) we have for any $\tau \in [\tau_\varepsilon^i, \tau_\varepsilon^{i+1}]$,

$$\begin{aligned} |\bar{z}(\tau) - z_\varepsilon(\tau)| \leq |\bar{z}(\tau) - \eta(\tau)| + |\eta(\tau) - \eta_i| + |\eta_i - \bar{z}_\varepsilon(\tau_\varepsilon^i)| + |\bar{z}_\varepsilon(\tau_\varepsilon^i) - \bar{z}_\varepsilon(\tau)| + \\ + |\bar{z}_\varepsilon(\tau) - z_\varepsilon(\tau)| \leq |\bar{z}_\varepsilon(\tau) - z_\varepsilon(\tau)| + \varepsilon S_\varepsilon[2P + cTe^{cT}P + cT^2e^{cT}\tilde{\beta}(\frac{T}{\varepsilon}, S_\varepsilon)] \end{aligned} \tag{3.4.7}$$

Now we will give an estimate of $|\bar{z}_\varepsilon(\tau) - z_\varepsilon(\tau)|$.

$$\begin{aligned} & |\bar{z}_\varepsilon(\tau) - z_\varepsilon(\tau)| \\ \leq\ & |\bar{z}_\varepsilon(\tau_\varepsilon^i) - z_\varepsilon(\tau_\varepsilon^i)| + \varepsilon \int_{\tau_\varepsilon^i}^{\tau} |f(y_\varepsilon(s), z_\varepsilon(\tau_\varepsilon^i), u_\varepsilon(s)) - f(y_\varepsilon(s), z_\varepsilon(s), u_\varepsilon(s))|\, \mathrm{d}s \\ \leq\ & |\bar{z}_\varepsilon(\tau_\varepsilon^i) - z_\varepsilon(\tau_\varepsilon^i)| + \varepsilon L \int_{\tau_\varepsilon^i}^{\tau} |z_\varepsilon(\tau_\varepsilon^i) - z_\varepsilon(s))|\, \mathrm{d}s \\ \leq\ & |\bar{z}_\varepsilon(\tau_\varepsilon^i) - z_\varepsilon(\tau_\varepsilon^i)| + \frac{1}{2}\varepsilon^2 S_\varepsilon^2 PL, \end{aligned}$$

4. cf for instance Theorem 5.3.1 in [7].

(because $|z_\varepsilon(\tau_\varepsilon^i) - z_\varepsilon(s))| \leq \varepsilon(s - \tau_\varepsilon^i)P$.)

So, denoting

$$d_\varepsilon^i(\tau) = \max_{\tau_\varepsilon^i \leq s \leq \tau} |\bar{z}_\varepsilon(s) - z_\varepsilon(s)|,$$

we have

$$d_\varepsilon^i(\tau_\varepsilon^{i+1}) \leq d_\varepsilon^i(\tau_\varepsilon^i) + (\varepsilon S_\varepsilon)^2 LP.$$

By iteration we obtain

$$d_\varepsilon^i(\tau_\varepsilon^{i+1}) \leq LP(\varepsilon S_\varepsilon)^2 i \leq LP(\varepsilon S_\varepsilon)^2(1 + \frac{T}{\varepsilon S_\varepsilon}) \leq LP(T + \varepsilon S_\varepsilon)\varepsilon S_\varepsilon.$$

Hence

$$\max_{\tau \in [0, \frac{T}{\varepsilon}]} |z_\varepsilon(\tau) - \bar{z}_\varepsilon(\tau)| \leq \max_{i \in I_\varepsilon} d_\varepsilon^i(\tau_\varepsilon^{i+1}) \leq LP(T + \varepsilon S_\varepsilon)\varepsilon S_\varepsilon. \tag{3.4.8}$$

Then estimates (3.4.7) and (3.4.8) give

$$|\bar{z}(\tau) - z_\varepsilon(\tau)| \leq LP(T + \varepsilon S_\varepsilon)\varepsilon S_\varepsilon + \varepsilon S_\varepsilon[2P + cTe^{cT}P + cT^2 e^{cT}\tilde{\beta}(\frac{T}{\varepsilon}, S_\varepsilon)].$$

By defining

$$\mu(T, \varepsilon) := LP(T + \varepsilon S_\varepsilon)\varepsilon S_\varepsilon + \varepsilon S_\varepsilon[2P + cTe^{cT}P + cT^2 e^{cT}\tilde{\beta}(\frac{T}{\varepsilon}, S_\varepsilon)].$$

Now recalling the expression (3.3.25) of $\tilde{\beta}$ we deduce that

$$\varepsilon S_\varepsilon \tilde{\beta}(\frac{T}{\varepsilon}, S_\varepsilon) = 2\varepsilon S_\varepsilon \beta(S_\varepsilon) + 2P\frac{\frac{T}{\varepsilon}}{\frac{T}{\varepsilon} + S_\varepsilon}\varepsilon S_\varepsilon = 2\varepsilon S_\varepsilon \beta(S_\varepsilon) + 2P\frac{\varepsilon S_\varepsilon}{1 + \frac{\varepsilon S_\varepsilon}{T}}.$$

Therefore,

$$\varepsilon S_\varepsilon \tilde{\beta}(\frac{T}{\varepsilon}, S_\varepsilon) \to 0,$$

as $\varepsilon \to 0$. Hence μ is continuous and $\mu(T, \varepsilon) \to 0$ as $\varepsilon \to 0$. So (3.2.2) holds true.

The proof is complete.

3.5 Examples

We first show a case where the limit set field $\bar{F}(y_0, z)$ depends on the initial value y_0.

3.5.1 Example

Let us consider the following control system with $m = 2$ and $n = 1$

$$\begin{cases} \dot{z}_\varepsilon(t) = u(t)|y_\varepsilon(t)|^2 - z_\varepsilon(t) \\ \varepsilon \dot{y}_\varepsilon(t) = Ay_\varepsilon(t), \end{cases} \tag{3.5.1}$$

where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $y_\varepsilon(0) = (y_1(0), y_2(0))$, $z_\varepsilon(0) = z_0 \in \mathbb{R}^n$ and $U = [0, 1]$

One can easily check that $M \times N = B \times [-1, 1] \subset \mathbb{R}^2 \times \mathbb{R}$ is invariant by the system (3.5.1) for every $\varepsilon > 0$.

From the last equation we get

$$\begin{cases} y_{\varepsilon,1}(t) = y_1(0)\cos\frac{t}{\varepsilon} - y_2(0)\sin\frac{t}{\varepsilon} \\ y_{\varepsilon,2}(t) = y_1(0)\sin\frac{t}{\varepsilon} + y_2(0)\cos\frac{t}{\varepsilon}. \end{cases}$$

So we observe that $|y_\varepsilon(t)|^2$ is a constant equal to $|y_0|^2$ consequently we have

$$\bar{F}(y_0, z) = |y_0|^2[0,1] - z.$$

Of course this example is very elementary and we are able to make all the computations without our main result. This is just to underline the fact that even in very simpler case the limit dynamic $\bar{F}$ may depend on y_0.

3.5.2 Counter-example

And finally in this section we give a counter-example [5] showing that in general every solution of the limit differential inclusion cannot be approximated by the solutions of the SPCS (as it can be when the limit dynamics $\bar{F}$ does not depend on y_0 cf [46]).

$$\begin{cases} \dot{z}_\varepsilon(t) = y_{1,\varepsilon}(t) \\ \varepsilon\dot{y}_{1,\varepsilon}(t) = \max\{1 - y_{2,\varepsilon}(t), 0\}u_\varepsilon(t) \\ \varepsilon\dot{y}_{2,\varepsilon}(t) = 2 - y_{2,\varepsilon}(t), \end{cases} \tag{3.5.2}$$

where $u_\varepsilon(t) \in [-1,1]$, $z_\varepsilon(0) = z$, $y_\varepsilon(0) = (y_1, 0)$.

As $y_2(0) = 0$, from the last equation it follows

$$y_{2,\varepsilon}(t) = 2(1 - e^{\frac{-t}{\varepsilon}}).$$

Therefore we have

$$\varepsilon\dot{y}_{1,\varepsilon}(t) = \max\{2e^{\frac{-t}{\varepsilon}} - 1, 0\}u_\varepsilon(t).$$

Thus

$$y_{1,\varepsilon}(t) = \begin{cases} y_1 + \int_0^t 2(1 - e^{\frac{-\tau}{\varepsilon}})u_\varepsilon(\tau)\,\mathrm{d}\tau, & \text{if } \ t \le \varepsilon\ln 2 \\ y_{1,\varepsilon}(\varepsilon\ln 2), & \text{if } \ t \ge \varepsilon\ln 2\ . \end{cases}$$

Hence

$$F_s(y_1, u_\varepsilon) = \bigcup_{u_\varepsilon(\cdot)} (\frac{1}{s}\int_0^s y_{1,\varepsilon}(t)\,\mathrm{d}t).$$

By taking, the "extremal" values of the control $u(\cdot) = -1$ and $u(\cdot) = +1$, one can obtain that

$$\bar{F}(y_0, z) = [y_1 - (1 + ln2), y_1 + 1 + ln2].$$

Fix $t > 0$, for sufficiently small $\varepsilon > 0$ we have

$$y_{1,\varepsilon}(t) = y_{1,\varepsilon}(\varepsilon ln2) \to_{\varepsilon\to 0} y_1.$$

Hence $z_\varepsilon(t)$ converge to ty_1. So any limit of $z_\varepsilon(t)$ is a straight line. While there are many solutions of

$$\dot{z}(t) \in \bar{F}(y_0, z(t)) = [y_1 - (1 + ln2), y_1 + 1 + ln2]$$

5. We would like to thank Pierre Cardaliaguet who brought our attention to this counter example.

which are not straight lines and consequently cannot be approximated by solutions to SPCS.

Acknowledgements. The work is partially supported by the French National Research Agency ANR-10-BLAN 0112 and the ITN - Marie Curie Grant n. 264735-SADCO.

Chapter 4

Nonexpansivity condition and some generalizations

Hayk Sedrakyan [1]

4.1 Introduction

In Chapter 3 (cf also [63]) we have considered weakly coupled singularly perturbed control system and investigated the behaviour of trajectories when the small perturbation parameter tends to zero by using the *averaging method* motivated by [46] and a recent result on nonexpansive control [62]. In the literature this problem has been usually considered under conditions ensuring that the limit dynamic is independent to the initial condition of the fast system (cf [46]). The main novelty of our averaging approach used in Chapter 3 lies in the fact that the limit dynamic may depend on the initial condition of the fast system. Our main result says that the limit trajectories of weakly coupled singularly perturbed control system are solutions to the corresponding differential inclusion [63]. But in contrast to results of [46] and [43], the trajectories of the corresponding differential inclusion do not approximate the solution of weakly coupled singularly perturbed control system. This fact was illustrated in Subsection 3.5.2 by discussing a corresponding counter-example. Moreover, another example was given in Subsection 3.5.1 to emphasize the fact that the limit field may depend on the initial value of the fast motion.

In this Chapter, we will generalise the results of [63] by considering a new nonexpansivity condition with a corresponding norm denoted by Δ. In order to illustrate the difference, let us notice that, when in particular $\Delta(x-y) = |x-y|$ (the Euclidean norm), our new nonexpansivity condition is becoming the nonexpansivity condition (A3) of Chapter 3.

Moreover, in Section 4.4 we consider an example where the new nonexpansity condition is satisfied but the nonexpansivity condition (A3) of Chapter 3 does not hold true.

We consider the following weakly coupled singularly perturbed control system on the

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bounded time interval $[0,T]$.

$$\begin{cases} \dot{z}_\varepsilon(t) = f(y_\varepsilon(t), z_\varepsilon(t), u_\varepsilon(t)), \;\; u_\varepsilon(t) \in U \\ \varepsilon \dot{y}_\varepsilon(t) = g(y_\varepsilon(t), u_\varepsilon(t)) \\ z_\varepsilon(0) = z_0 \\ y_\varepsilon(0) = y_0, \end{cases} \tag{4.1.1}$$

where $z_\varepsilon(\cdot) : [0,T] \to \mathbb{R}^m$ is the slow motion, $y_\varepsilon(\cdot) : [0,T] \to \mathbb{R}^n$ is the fast motion, $u_\varepsilon(\cdot)$ is the control function taking values in a given nonempty metric space U and $z_\varepsilon(0) = z_0 \in \mathbb{R}^m$, $y_\varepsilon(0) = y_0 \in \mathbb{R}^n$ are the initial values.

Let us consider the following system

$$\begin{cases} \dot{y}(t) = g(y(t), u(t)) \\ y(0) = y_0, \end{cases} \tag{4.1.2}$$

the unique solution of which is denoted by $t \mapsto y(t, y_0, u)$ and for any $z \in \mathbb{R}^m$ define the following set-valued map

$$F(S, y_0, z) \doteq cl \bigcup_{u \in U} \{ \frac{1}{S} \int_0^S f(y(s, y_0, u), z, u(s)) \, \mathrm{d}s \}.$$

Under suitable assumptions, it is possible to prove that $F(S, y_0, z)$ converge (when $S \to \infty$) to some $\bar{F}(y_0, z)$. As we have already mentioned the main aim of our work consists in investigating a case where the limit equation

$$\begin{cases} \dot{z}(t) \in \bar{F}(z(t), y_0) \\ z(0) = z_0, \end{cases} \tag{4.1.3}$$

could depend on y_0.

4.2 Main results

Denote by $\mathcal{U}$ the set of measurable controls from $\mathbb{R}_+$ to U.

We denote by $G(y_0) = \{y(t, y_0, u), t \geq 0, u \in \mathcal{U}\}$ the reachable set, i.e. the set of states, that can be reached starting from y_0 by trajectories of (4.1.2).

We make the following

Assumptions.

(A1) f, g are Lipschitz continuous with Lipschitz constant smaller or equal to $L > 0$.

(A2) there exists a compact set $M \times N \subset \mathbb{R}^n \times \mathbb{R}^m$ which is invariant by (4.1.1) for all $\varepsilon > 0$, namely if $(y_0, z_0) \in M \times N$, then for every control $u_\varepsilon(\cdot)$ the corresponding solution to (4.1.1) satisfies $(y_\varepsilon(t), z_\varepsilon(t)) \in M \times N$ for all $t \geq 0$.

(A3) (Nonexpansivity condition) For any $y_1, y_2 \in M$, $z \in N$ and $u \in U$ there exists $v \in U$ and $C > 0$ such that

$$\begin{cases} \langle \nabla(\Delta^2(y_1 - y_2)), g(y_1, u) - g(y_2, v)) \rangle \leq 0 \\ |f(y_1, z, u) - f(y_2, z, v)| \leq C |y_1 - y_2|, \end{cases}$$

where Δ is a norm such that $(x, y) \mapsto \Delta^2(x - y)$ is continuously differentiable. Notice that $\Delta(\cdot)$ is equivalent to $|\cdot|$, we denote by $l > 0$ the constant such that $\frac{1}{l}|x| \leq \Delta(x) \leq l|x|, \;\forall x \in \mathbb{R}^n$.

(A4) For all $y_1, y_2 \in M$, $z \in N$, $T > 0$, $u \in U$, the set

$$\{(g(y_1, u), g(y_2, v), 0)) \,|v \in U, |f(y_1, z, v) - f(y_2, z, u)| \leq C|y_1 - y_2|\}$$

is closed and convex.

Remark 4.2.1. *Observe that from **(A1)** and **(A2)**, we deduce that some $P \geq 0$ exists such that for all $(y, z, u) \in M \times N \times U$ we have $|f(y, z, u)| + |g(y, u)| \leq P$.*

*When f does not depend on its third variable or when the nonexpansivity condition **(A3)** is fulfilled with $u = v$, then the condition **(A4)** is implied by **(A1)**, **(A3)** and the following simpler assumption*

***(A4')** For all $y \in M$, the set $g(y, U)$ is closed and convex.*

We are now in position to state our main result

Theorem 4.2.2. *If (A1)-(A4) hold true. There exist a Lipschitz set-valued map $\bar{F} : M \times N \to \mathbb{R}^m$ and a continuous function $\mu : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ with $\mu(\cdot, 0) = 0$ such that the following approximation property of the slow motion holds true.*

For any $T > 0$, $\varepsilon > 0$, $(y_0, z_0) \in M \times N$ and for any solution $(y_\varepsilon(\cdot), z_\varepsilon(\cdot))$ to (4.1.1), there exists a solution $\bar{z}(\cdot)$ to (4.1.3) which satisfies

$$\max_{t \in [0,T]} |z_\varepsilon(t) - \bar{z}(t)| \leq \mu(T, \varepsilon). \tag{4.2.1}$$

4.3 On the limit differential inclusion

The main result of this section concerns the construction of a limit differential inclusion driven by $\bar{F}$ which is given by the following

Proposition 4.3.1. *If (A1)-(A4) hold, then there exists a Lipschitz continous set-valued map $\bar{F} : \mathbb{R}^n \times \mathbb{R}^m \rightsquigarrow \mathbb{R}^m$ with convex, compact images and a nonincreasing function $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\lim_{s \to \infty} \beta(S) = 0$, such that*

$$d_H[\bar{F}(y_0, z), \bar{co}\{\frac{1}{S} \int_0^S f(y(\tau, y_0, u(\cdot)), z, u(\tau))\, \mathrm{d}\tau,\ u(\cdot) \in \mathcal{U}\}] \leq \beta(S), \tag{4.3.1}$$

where the slow state $z \in N$ is fixed, $y(\tau, y_0, u(\cdot))$ denotes the solution of (4.1.2) and $d_H[\cdot, \cdot]$ denotes the Hausdorff distance.

Proof. Let $s > 0$ and $p \in B \subset \mathbb{R}^m$. Similar to Section 3.3 we define the cost between times m and $m + s$,

$$\gamma_{m,s}(y_0, z, p, u(\cdot)) = \frac{1}{s} \int_m^{m+s} \langle f(y(\tau, y_0, u(\cdot)), z, u(\tau)), p\rangle \, \mathrm{d}\tau. \tag{4.3.2}$$

Denote

$$F_{m,s}(y_0, z, p) = \inf_{u(\cdot) \in \mathcal{U}} \gamma_{m,s}(y_0, z, p, u(\cdot)). \tag{4.3.3}$$

Set

$$\gamma_s(y_0, z, p, u) \doteq \gamma_{0,s}(y_0, z, p, u)$$

and

$$F_s(y_0, z, p) = F_{0,s}(y_0, z, p).$$

We are interested in the limit behaviour of $F_s(y_0, z, p)$, when $s \to \infty$.
Let us introduce the following notations

$$F^-(y_0, z, p) \doteq \lim_{s\to\infty} \inf F_s(y_0, z, p) \tag{4.3.4}$$

$$F^+(y_0, z, p) \doteq \lim_{s\to\infty} \sup F_s(y_0, z, p) \tag{4.3.5}$$

The Proposition 4.3.1 follows from some auxilliary Lemmas and Proposition we will state now.

Lemma 4.3.2. *For every $m_0 \in \mathbb{R}_+$ and every $p \in B$ we have*

$$\sup_{s>0} \inf_{m\le m_0} F_{m,s}(y_0, z, p) \ge F^+(y_0, z, p) \ge F^-(y_0, z, p) \ge \sup_{s>0} \inf_{m\ge 0} F_{m,s}(y_0, z, p).$$

Proof. The proof is very similar to the one of Lemma 3.3.2. □

Proposition 4.3.3. *Assume that (A1)–(A4) hold true. Then, for all $y_1, y_2 \in M$, $z \in N$, $T > 0$, $u(\cdot) \in \mathcal{U}$, there exists $v(\cdot) \in \mathcal{U}$ and a constant $K > 0$ such that*

$$\Delta(y(t, y_1, u(\cdot)) - y(t, y_2, v(\cdot))) \le \Delta(y_1 - y_2),\ \forall t \in [0, T], \tag{4.3.6}$$

and for almost every $t \in [0, T]$,

$$|f(y(t, y_1, u(\cdot)), z, u(t)) - f(y(t, y_2, v(\cdot)), z, v(t))| \le K|y_1 - y_2|. \tag{4.3.7}$$

Proof. Fix $y_1, y_2 \in M$, $z \in N$, $T > 0$ and $u \in U$. We consider the map

$$\begin{aligned} \phi(t, x, y, l) \quad = \quad & \{(g(x, u(t)), g(y, v), 0)) \,|v \in U, \\ & |f(y, z, v) - f(x, z, u(t))| \le K|x - y|\}\,. \end{aligned}$$

which has compact convex values and which is upper semicontinuous with respect to (x, y, l) and measurable in t with closed convex values due to assumptions (A1)–(A4). Observe that $\phi(t, x, y, l)$ does not depend on l. From the measurable Viability Theorem [32], assumption (A3) implies that the epigraph of the function $(x, u) \mapsto \Delta(x-y)$ is viable for the differential inclusion

$$(x'(t), y'(t), l'(t)) \in \phi(t, x(t), y(t), l(t)),\ for\ a.e.\ t \ge 0, \tag{4.3.8}$$

Therefore, starting from $(y_1, y_2, \Delta(y_1 - y_2))$ and noticing that $l(t)$ is constant, there exists a solution $(x(.), y(.), l(.))$ of (4.3.8) satisfying

$$\Delta(x(t) - y(t)) \le l(t) = \Delta(y_1 - y_2),\ \forall t \ge 0.$$

From one hand we have clearly $x(.) = y(., y_1, u)$ and from the other hand, by Filippov's measurable selection Theorem there exists a control $v \in \mathcal{U}$ such that $y(.) = y(., y_2, v)$. So (4.3.6) holds true. Moreover, from the very definition of ϕ, we obtain (4.3.7). The proof is complete. □

Now we will use the above result to build the map $\bar{F}$. For doing it we introduce the map

$$F(y_0, z, p) = \sup_{s>0} \inf_{m\ge 0} F_{m,s}(y_0, z, p). \tag{4.3.9}$$

the regularity of which is obtained in the following

Lemma 4.3.4. *There exists a constant $c > 0$ such that for all $(y_1, z_1), (y_2, z_2) \in M \times N$, all $p \in B$ and all $s > 0$, we have*

$$|F_s(y_1, z_1, p) - F_s(y_2, z_2, p)| \leq c(|y_1 - y_2| + |z_1 - z_2|)|p|.$$

Proof. Fix $\eta > 0$, there exists $u(\cdot) \in \mathcal{U}$ such that

$$F_s(y_2, z_1, p) \leq \frac{1}{s} \int_0^s \langle f(y(\tau, y_2, u(\cdot)), z_1, u(\tau)), p\rangle \, \mathrm{d}\tau \leq F_s(y_2, z_1, p) + \eta. \tag{4.3.10}$$

Using Proposition 4.3.3, associated with $u(\cdot)$there exists $v(\cdot) \in U$ such that (4.3.7) holds true.

Thus (4.3.10) yields

$$\begin{aligned} &F_s(y_1, z_1, p) - F_s(y_2, z_1, p) \leq \\ \frac{1}{s} \int_0^s &\langle f(y(\tau, y_1, v(\cdot)), z_1, v(\tau)), p\rangle \quad - \langle f(y(\tau, y_2, u(\cdot)), z_1, u(\tau)), p\rangle \, \mathrm{d}\tau + \eta \\ &\leq K|y_1 - y_2||p| + \eta. \end{aligned}$$

The Lipschitz continuity of $f(y, \cdot, u)$ implies that for any $y_1, y_2 \in M$, $z_1, z_2 \in N$

$$|F_s(y_2, z_1, p) - F_s(y_2, z_2, p)| \leq L|z_1 - z_2||p|.$$

So interchanging the role of (y_1, z_1) and (y_2, z_2) and taking into account that η is arbitrary, our claim follows from the triangular inequality with $c = L + K$. $\square$

Define now for each $m \geq 0$ the set $G^m(y_0) = \{y(s, y_0, u), s \leq m, u \in \mathcal{U}\}$, (the set of states which can be reached from y_0 before time m), thus $G(y_0) = \bigcup_{m \geq 0} G^m(y_0)$.

Lemma 4.3.5. *For any $\varepsilon > 0$ there exists $m_0 > 0$, such that for any $x \in G(y_0)$ there exists $x_1 \in G^{m_0}(y_0)$, such that $|x - x_1| \leq \varepsilon$.*

Proof. The proof follows from the boundedness of $G(y_0)$ (cf Lemma 3.9 in [62]). $\square$

Lemma 4.3.6. *For any $(y_0, z) \in M \times N$ and $|p| \leq 1$*

$$\lim_{s \to \infty} F_s(y_0, z, p) = F(y_0, z, p).$$

Proof. The proof is very similar to the one of Lemma 3.3.6 using Lemma 4.3.2, Lemma 4.3.4 and Lemma 4.3.5 instead of Lemma 3.3.2, Lemma 3.3.4, Lemma 3.3.5. $\square$

As consequence of Lemma 4.3.4 and Lemma 4.3.6 we obtain

Corollary 4.3.7. *$F(y_0, z, p)$ is Lipschitz continuous with respect to y_0 and z.*

Now let us proceed the proof of Proposition 4.3.1.

End of the proof of Proposition 4.3.1.
In view of Lemma 4.3.4 and Lemma 4.3.6 we have obtained that

$$\inf_{u(\cdot) \in \mathcal{U}} \frac{1}{s} \int_0^s \langle f(y(\tau, y_0, u(\cdot)), z, u(\tau)), p\rangle \, \mathrm{d}\tau \to_{s \to \infty} F(y_0, z, p). \tag{4.3.11}$$

Let us define

$$\bar{F}(y_0, z) = \{\xi \in \mathbb{R}^m; \ \langle \xi, p\rangle \leq F(y_0, z, p), \ \forall p \in \mathbb{R}^m\}.$$

The set-valued map $\bar{F}(\cdot,\cdot)$ has clearly compact convex values which are bounded by P. By QuSe we have that $\bar{F}$ is Lipschitz continuous with Lipschitz constant less or equal to c.

By Lemma 4.3.4 the family $\{F_s(\cdot,\cdot,p)\}_{p\in B}$ is equi-continuous on $M\times N$. Using Arzela-Ascoli theorem and by Lemma 4.3.6 we deduce that $F_s(y_0,z,p)$ converge uniformly to $F(y_0,z,p)$ on $M\times N$, when $s\to\infty$.

A straightforward verification shows that

$$\beta(s,p)=\sup_{(y_0,z)\in M\times N}|F_s(y_0,z,p)-F(y_0,z,p)|.$$

is homogeneous and continuous with respect to p. By replacing $\beta(s,p)$ by $\sup_{\sigma\in(0,s]}\beta(\sigma,p)$ we obtain that $\beta(s,p)$ is nonincreasing with respect to s.

Therefore by taking $\beta(s)=\sup_{p\in B}\beta(s,p)$ we obtain in view of Lemma 4.3.6 that $\beta(s)\to 0$, as $s\to\infty$ and β is nonincreasing.

We may proceed and complete the proof of (4.3.1) as it is done in [63].

The proof of Proposition 4.3.1 is complete. □

Once $\bar{F}$ is obtained, we need a result more precise than Proposition 4.3.1.

Proposition 4.3.8. *If (A1)-(A4) hold true, then for all $s>0$ and $m\geq 1$*

$$d_H[\bar{F}(y_0,z),\bar{co}\{\frac{1}{s}\int_m^{s+m} f(y(\tau,y_0,u(\cdot)),z,u(\tau))\,\mathrm{d}\tau,\ u(\cdot)\in\mathcal{U}\}]\leq 2\beta(s)+\frac{2m}{m+s}P. \tag{4.3.12}$$

Proof. The proof is very similar to the one of Proposition 3.3.1. □

We denote

$$\tilde{\beta}(m,s)\doteq 2\beta(s)+\frac{2m}{m+s}P, \tag{4.3.13}$$

which is increasing in m.

Consequently we have that for any $m\geq 1$, $s>0$, $y_0\in M$, $z\in N$

$$d_H[\bar{F}(y_0,z),\bar{co}\{\frac{1}{s}\int_m^{s+m} f(y(\tau,y_0,u(\cdot)),z,u(\tau))\,\mathrm{d}\tau,\ u(\cdot)\in\mathcal{U}\}]\leq\tilde{\beta}(m,s). \tag{4.3.14}$$

Remark 4.3.9. *The proof of the main result (Theorem 4.2.2) is very similar to the one of the Section 3.4, using Proposition 4.3.8 and* (4.3.14) *instead of Proposition 3.3.8.*

4.4 Example

The crucial role for the approximation of the slow motion plays the nonexpansivity condition (A3). In order to emphasize this fact and to show the differences between this Chapter and Chapter 3, in this section we give an example, where condition (A3) is satisfied (with $a>0$) for the norm

$$\Delta(y_1,y_2)=\sqrt{|y_1|^2+\frac{1}{a}|y_2|^2}, \tag{4.4.1}$$

but the nonexpansivity condition of (A3) of Chapter 3 is not satisfied.

4.4.1 Example.

Let us consider the following control system with $m = 2$ and $n = 1$

$$\begin{cases} \dot{z}_\varepsilon(t) = u(t)\Delta^2(y_\varepsilon(t)) - z_\varepsilon(t) \\ \varepsilon \dot{y_\varepsilon}(t) = Ay_\varepsilon(t), \\ y_\varepsilon(0) = y_0 \\ \mathrm{z}_\varepsilon(0) = z_0. \end{cases} \tag{4.4.2}$$

where $A = \begin{pmatrix} 0 & 1 \\ -a & 0 \end{pmatrix}$, $a > 0$, $t \in [0, T]$, $T > 0$, $y_0 = \Big(-\dfrac{y_1(0)}{\sqrt{a}}, y_2(0) \Big)$, $z_0 \in \mathbb{R}$, $U = [0, 1]$ and $\Delta^2(y_\varepsilon(t)) \doteq \Delta^2(y_{\varepsilon,1}(t), y_{\varepsilon,2}(t))$.

Proof. Let us notice that from the last equation of (4.4.2) it follows that

$$\begin{cases} y_{\varepsilon,1}(t) = -\dfrac{y_1(0)}{\sqrt{a}} \cos \frac{t\sqrt{a}}{\varepsilon} + \dfrac{y_2(0)}{\sqrt{a}} \sin \frac{t\sqrt{a}}{\varepsilon} \\ y_{\varepsilon,2}(t) = y_1(0) \sin \frac{t\sqrt{a}}{\varepsilon} + y_2(0) \cos \frac{t\sqrt{a}}{\varepsilon}. \end{cases} \tag{4.4.3}$$

From (4.4.1) and (4.4.3) we deduce that $\Delta^2(y_\varepsilon(t))$ is a constant equal to $\Delta^2(y_0)$, consequently we obtain that

$$\bar{F}(y_0, z) = \Delta^2(y_0)[0, 1] - z.$$

Notice that, if we define $B^{\Delta^2} = \{y \mid \Delta^2(y) \leq 1\}$, one can easily check that $M \times N = B^{\Delta^2} \times [-1, 1] \subset \mathbb{R}^2 \times \mathbb{R}$ is invariant by the system (4.4.2) (using that $\Delta^2(y_\varepsilon(t)) = \Delta^2(y_0)$).

Now, let us prove that assumption (A3) is satisfied for the system (4.4.2).

We take $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and denote by $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = w \doteq x - y$.

We have that

$$\Delta(w_1, w_2) = \sqrt{|w_1|^2 + \frac{1}{a}|w_2|^2}.$$

Therefore

$$\nabla(\Delta^2(x - y)) = \nabla(\Delta^2(w)) = \nabla(w_1^2 + \frac{1}{a}w_2^2) = (2w_1, \frac{2}{a}w_2).$$

Thus

$$\langle \nabla(\Delta^2(x - y)), g(x) - g(y) \rangle = 2 \langle \begin{pmatrix} x_1 - y_1 \\ \frac{1}{a}(x_2 - y_2) \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -a & 0 \end{pmatrix} \begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \end{pmatrix} \rangle =$$

$$= 2 \langle \begin{pmatrix} x_1 - y_1 \\ \frac{1}{a}(x_2 - y_2) \end{pmatrix}, \begin{pmatrix} x_2 - y_2 \\ -a(x_1 - y_1) \end{pmatrix} \rangle = 2(x_1 - y_1)(x_2 - y_2) - 2(x_1 - y_1)(x_2 - y_2) = 0.$$

Hence assumption (A3) is satisfied, as

$$\langle \nabla(\Delta^2(x - y)), g(x) - g(y) \rangle \leq 0.$$

Let us now prove that the nonexpansivity condition (A3) of Chapter 3 does not hold true for our example. We proceed by a contradiction argument, assume it holds true, therefore for any $a > 0$ and any $y_1, y_2, x_1, x_2 \in M$,

$$\langle g \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - g \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 - x_1 \\ y_2 - x_2 \end{pmatrix} \rangle \leq 0.$$

Therefore

$$\langle \begin{pmatrix} 0 & 1 \\ -a & 0 \end{pmatrix} \begin{pmatrix} y_1 - x_1 \\ y_2 - x_2 \end{pmatrix}, \begin{pmatrix} y_1 - x_1 \\ y_2 - x_2 \end{pmatrix} \rangle \leq 0.$$

Hence we deduce that

$$(y_2 - x_2)(y_1 - x_1)(1 - a) \leq 0.$$

Taking $x_2 = 2, y_2 = 1, y_1 = 1, x_1 = a$ we get a contradiction. Therefore, the nonexpansivity condition (A3) of Chapter 3 is not satisfied in this case.

□

Part II

Chapter 5

Stable Representation of Convex Hamiltonians

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Published in J. Nonlinear Analysis, TMA, 100 (2014), pp. 30-42.

Abstract. Existence and uniqueness of solutions to a Hamilton-Jacobi equation

$$v_t + H(t, x, v_x) = 0, \quad v(0, \cdot) = \varphi(\cdot)$$

with H convex with respect to the last variable can be proved by associating to H either a Calculus of Variations or an optimal control problem. The data of the new problem should be so that its Hamiltonian coincides with H and should also inherit appropriate regularity properties of the Hamiltonian. In other words, H can be represented by functions describing an optimisation problem. In this paper we provide further developments of representation theorems from [65]. In particular, our results imply stability of representations with respect to Hamiltonians. We apply them to study existence of solutions to the Hamilton-Jacobi equation with $t-$measurable Hamiltonian.

Keywords: Hamilton-Jacobi equations, optimal control, representation of Hamiltonians, sensitivity.

AMS Mathematics Subject Classification 2010. Primary 26E25,49L25. Secondary 34A60.

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5.1 Introduction

Consider a Hamiltonian $H : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ that is convex in the last variable and the Hamilton-Jacobi equation

$$- v_t + H(t, x, -v_x) = 0, \quad v(T, \cdot) = \varphi(\cdot). \tag{5.1.1}$$

Such way of stating the Cauchy problem is more convenient for our purposes than the usual initial value problem. Replacing t by $T-t$ and redefining H, (5.1.1) may be reduced to the equation mentioned in the abstract.

Let $H^*(t, x, \cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be the Fenchel conjugate of $H(t, x, \cdot)$ and consider the Calculus of Variations problem

$$v(t_0, x_0) = \inf\{\varphi(x(T)) + \int_{t_0}^{T} H^*(t, x(t), x'(t))dt \, : \, x \in W^{1,1}([t_0, T], \; x(t_0) = x_0\}.$$

Under appropriate assumptions, v is the unique (viscosity) solution of (5.1.1), see for instance [18]. In particular, regularity of the solution v can be studied in a standard way by using regularity of H^* . It may happen however that H^* takes infinite values. Then the investigation of regularity of v and proofs of uniqueness of solutions to the above Hamilton-Jacobi equation may become difficult.

A natural question arises : can we associate to H mappings $f : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ and $l : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}$ inheriting Lipschitz type regularity properties of H and such that $f(t, x, U)$ is equal to the domain of $H^*(t, x, \cdot)$ and

$$H(t, x, p) = \max_{u \in U}(\langle p, f(t, x, u)\rangle - l(t, x, u)), \tag{5.1.2}$$

where U is a compact subset of a finite dimensional space.

That is H is equal to the Hamiltonian of a Bolza optimal control problem:

$$V(t_0, x_0) = \inf \left\{ \varphi(x(T)) + \int_{t_0}^{T} l(t, x(t), u(t))dt \Big| \; (x, u) \in \mathcal{S}(t_0, x_0) \right\}. \tag{5.1.3}$$

Here, $\mathcal{S}(t_0, x_0)$ denotes the set of all trajectory-control pairs of the control system

$$\begin{cases} x'(t) &= f(t, x(t), u(t)), \quad u(t) \in U \qquad \text{a.e.} \\ x(t_0) &= x_0 \, . \end{cases} \tag{5.1.4}$$

Under appropriate assumptions, V is the unique solution to (5.1.1), cf. [18].

When working with control systems it is natural to require from f to be so that to every measurable control $u : [t_0, T] \to U$ corresponds a unique state-trajectory $x(\cdot)$ defined on $[t_0, T]$. This is guaranteed by the Lipschitzianity and the sublinear growth of f with respect to x.

Denote by $F(t, x)$ the domain of $H^*(t, x, \cdot)$. Then $F(t, x)$ is convex and it can be parameterized in the way preserving some regularity properties of F, see for instance [9].

A couple (f, l) satisfying (5.1.2) is called in [65] a faithful representation of the Hamiltonian H whenever f enjoys Lipschitz continuity with respect to x. The proof of Lipschitz continuity of $F(t, \cdot)$ (cf. [65, Theorem 3.2]) is based on a fixed point theorem and a contradiction argument. The moduli of continuity of H with respect to x and p are assumed to be time independent. In [65] the interested reader can also find references and comments on the earlier literature concerning representation theorems.

The present paper is devoted to regularity of f, l when H is locally Lipschitz with respect to x. In particular we show that the Lipschitz constants of $f(t,\cdot,u)$, $l(t,\cdot,u)$ can be estimated using the Lipschitz constant of $H(t,\cdot,p)$. For this aim, we replace the fixed point argument of [65] by a much simpler in use separation theorem. This also allows us to investigate stability of representations with respect to Hamiltonians.

The representation theorem is applied then to study existence of a lower semicontinuous solution to the Hamilton-Jacobi equation (5.1.1) for Hamiltonian H merely measurable with respect to time. Actually this solution is the value function of the associated Bolza problem. We do not investigate uniqueness of a solution here, though, once a solution is described via the value function, this can be done in the same vein as in [39].

Results on stability of representations are applied to study stability of solutions to a Hamilton-Jacobi equation under state constraints in [69].

The outline of the paper is as follows. In Section 5.2 we recall some notions and introduce some notations. In Section 5.3 we provide a representation theorem and in Section 5.4 we investigate stability of representations for Hamiltonians continuous with respect to time. Section 5.5 is devoted to the case of Lebesgue measurable in time Hamiltonians. In Section 5.6 existence of lower semicontinuous solutions to the Hamilton-Jacobi equation is proved.

5.2 Preliminaries and Notations

The notations B_R and $B(0,R)$ stand for the closed ball in $\mathbb{R}^n$ of center zero and radius $R \geq 0$ and $B := B_1$. We denote by $\langle p, v\rangle$ the scalar product of p, $v \in \mathbb{R}^n$. For a map $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ its domain is defined by

$$dom(\phi) \doteq \{x \in \mathbb{R}^n : \phi(x) < +\infty\}$$

and its epigraph $epi(\phi)$ by

$$epi(\phi) \doteq \{(x,a) : x \in \mathbb{R}^n,\ a \in \mathbb{R},\ a \geq \phi(x)\} \subseteq \mathbb{R}^{n+1}.$$

If ϕ is differentiable at $x \in \mathbb{R}^n$, then $\nabla\phi(x)$ states for its gradient at x.

Recall that the *normal cone* to a convex set $K \subset \mathbb{R}^n$ at $x \in K$ is defined by

$$N_K(x) \doteq \{p \in \mathbb{R}^n : \langle p, y - x\rangle \leq 0, \forall\, y \in K\}.$$

For a convex function $f : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$, $\partial f(x)$ denotes its subdifferential at $x \in dom(f)$ and f^* its Fenchel conjugate.

Recall that $(q,-1) \in N_{epi(f)}(x, f(x))$ if and only if $q \in \partial f(x)$.

Lemma 5.2.1 ([67], p.476)**.** *For any proper, lower semicontinuous, convex function $f : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$, one has $\partial f^* = (\partial f)^{-1}$ and $\partial f = (\partial f^*)^{-1}$. In the other words $p \in \partial f^*(v)$ is equivalent to $v \in \partial f(p)$ and is equivalent to $f(p) + f^*(v) = \langle v, p\rangle$.*

The following Lemma is a consequence of a more general result [66, Theorem 1].

Lemma 5.2.2 ([66])**.** *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be convex, lower semi-continuous, $x \in dom(f)$ and $(p,0) \in N_{epi(f)}(x, f(x))$. Then there exists a sequence $x_i \in \mathbb{R}^n$ such that $(x_i, f(x_i))$ converge to $(x, f(x))$ when $i \to \infty$ and for some $(p_i, q_i) \in N_{epi(f)}(x_i, f(x_i))$ we have $q_i < 0$ and (p_i, q_i) converge to $(p, 0)$. In particular, $\dfrac{p_i}{|q_i|} \in \partial f(x_i)$.*

Let $\varphi : \mathbb{R}^m \to \mathbb{R} \cup \{\pm\infty\}$, $x_0 \in dom(\varphi)$. The *superdifferential* and *subdifferential* of φ at x_0 are defined respectively by:

$$\partial_+\varphi(x_0) = \left\{ p \in \mathbb{R}^m \mid \limsup_{x \to x_0} \frac{\varphi(x) - \varphi(x_0) - \langle p, x - x_0 \rangle}{|x - x_0|} \leq 0 \right\},$$

$$\partial_-\varphi(x_0) = \left\{ p \in \mathbb{R}^m \mid \liminf_{x \to x_0} \frac{\varphi(x) - \varphi(x_0) - \langle p, x - x_0 \rangle}{|x - x_0|} \geq 0 \right\}.$$

The upper and lower directional derivatives of φ at x_0 in the direction $\bar{v} \in \mathbb{R}^m$ are defined respectively by

$$D_\downarrow\varphi(x_0)(\bar{v}) := \limsup_{h \to 0+,\, v \to \bar{v}} \frac{\varphi(x + hv) - \varphi(x)}{h},$$

$$D_\uparrow\varphi(x_0)(\bar{v}) := \liminf_{h \to 0+,\, v \to \bar{v}} \frac{\varphi(x + hv) - \varphi(x)}{h}.$$

Recall that $p \in \partial_+\varphi(x_0)$ if and only if for

$$\forall\ v \in \mathbb{R}^m, \langle p, v \rangle \geq D_\downarrow\varphi(x_0)(v)$$

and $p \in \partial_-\varphi(x_0)$ if and only if for

$$\forall\ v \in \mathbb{R}^m, \langle p, v \rangle \leq D_\uparrow\varphi(x_0)(v).$$

Let A be a metric space with the distance d and K be a subset of A. The distance from $x \in A$ to K is defined by

$$d(x, K) := \inf_{y \in K} d(x, y),$$

where we have set $d(x, \emptyset) = +\infty$.

Let $\{K_i\}_{i \geq 1}$ be a family of subsets of a metric space A. The subset

$$Limsup_{i \to \infty} K_i := \{x \in A : \lim\inf_{i \to \infty} d(x, K_i) = 0\}$$

is called the upper limit of the sequence K_i and the subset

$$Liminf_{i \to \infty} K_i := \{x \in A : \lim_{i \to \infty} d(x, K_i) = 0\}$$

is called its lower limit. A subset K is said to be the (set) limit of the sequence K_i if

$$K = Liminf_{i \to \infty} K_i = Limsup_{i \to \infty} K_i =: Lim_{i \to \infty} K_i.$$

For arbitrary subsets K, L of $\mathbb{R}^n$, the extended Hausdorff distance between K and L is defined by

$$\mathcal{H}(K, L) := max\{\sup_{x \in K} d(x, L), \sup_{x \in L} d(x, K)\} \in \mathbb{R} \cup \{+\infty\},$$

which may be equal to $+\infty$ when K or L is unbounded or empty.

It is well known that if K_i are subsets of a given compact set, then

$$K = Lim_{i \to \infty} K_i \quad \Leftrightarrow \quad \lim_{i \to \infty} \mathcal{H}(K_i, K) = 0.$$

Lemma 5.2.3 ([9], p.369)**.** *Let Ω denote the family of all nonempty closed convex sets in $\mathbb{R}^n$. Then the map $P : \mathbb{R}^n \times \Omega \rightsquigarrow \Omega$ defined by*

$$P(y, K) = K \cap B(y, 2d(y, K))$$

is Lipschitz with the Lipschitz constant 5, i.e. for all $K, L \in \Omega$ and $x, y \in \mathbb{R}^n$

$$\mathcal{H}(P(x, K), P(y, L)) \leq 5(\mathcal{H}(K, L) + |x - y|).$$

The support function $\sigma(K, \cdot) : \mathbb{R}^n \to \mathbb{R}$ of a nonempty, convex, compact set $K \subset \mathbb{R}^n$ is defined by

$$\sigma(K, p) := \max_{x \in K} \langle p, x \rangle,$$

for every $p \in \mathbb{R}^n$.

The subdifferential of $\sigma(K, \cdot)$ at $p \in \mathbb{R}^n$ is equal then to the set

$$\partial\sigma(K, p) := \{q \in \mathbb{R}^n : \sigma(K, p_1) - \sigma(K, p) \geq \langle q, p_1 - p \rangle, \forall\, p_1 \in \mathbb{R}^n\}.$$

The function $\sigma(K, \cdot)$ being Lipschitz, $\sigma(K, \cdot)$ is differentiable a.e. in $\mathbb{R}^n$.
Let $m(\partial\sigma(K, p))$ denote the element of $\partial\sigma(K, p)$ with the minimal norm.
It coincides then with $\nabla\sigma(K, p)$ at every $p \in \mathbb{R}^n$ where $\sigma(K, \cdot)$ is differentiable.

Definition 5.2.4. *Let Ω be the family of all nonempty convex compact subsets of $\mathbb{R}^n$. For any $K \in \Omega$ the Steiner point of K is defined by*

$$s_n(K) := \frac{1}{vol(B)} \int_B m(\partial\sigma(K, p))\, \mathrm{d}p,$$

where $vol(B)$ is the measure of the n-dimensional unit ball $B \subset \mathbb{R}^n$.

By [9, p.366], $s_n(\cdot)$ is Lipschitz in the Hausdorff metric with the Lipschitz constant n.

Definition 5.2.5. *For a map $H : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, H^* denotes the conjugate of H with respect to the third variable, i.e. for all $(t, x, v) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$*

$$H^*(t, x, v) := \sup_{p \in \mathbb{R}^n} \{\langle v, p \rangle - H(t, x, p)\} \in \mathbb{R} \cup \{+\infty\}$$

and $\partial_p H(t, x, \bar{p})$ denotes the subdifferential of $H(t, x, \cdot)$ at $\bar{p}$.

Proposition 5.2.6. *Let $t \in [0, T]$. If $H(t, \cdot, \cdot)$ is upper semicontinuous, then $H^*(t, \cdot, \cdot)$ is lower semicontinuous.*

Proof. Consider $(x_i, v_i) \to (x, v)$ such that

$$\liminf_{(y,w) \to (x,v)} H^*(t, y, w) = \lim_{i \to \infty} H^*(t, x_i, v_i).$$

If this lower limit is equal to $+\infty$, then

$$H^*(t, x, v) \leq \liminf_{(y,w) \to (x,v)} H^*(t, y, w).$$

Assume next that it is finite. Then for some $M \geq 0$ and all large i, $H^*(t, x_i, v_i) \leq M$. Fix any $p \in \mathbb{R}^n$. Thus for all large i,

$$\langle v, p \rangle - H(t, x, p) \leq \langle v_i, p \rangle - H(t, x_i, p) \leq M + 1.$$

Since p is arbitrary, $H^*(t,x,v) \leq M+1$. Let $\varepsilon > 0$ and consider $p \in \mathbb{R}^n$ such that

$$H^*(t,x,v) \leq \langle v,p\rangle - H(t,x,p) + \varepsilon.$$

By the upper semicontinuity assumption, for all large i,

$$\langle v,p\rangle - H(t,x,p) \leq \langle v_i,p\rangle - H(t,x_i,p) + \varepsilon \leq H^*(t,x_i,v_i) + \varepsilon.$$

Hence

$$H^*(t,x,v) \leq H^*(t,x_i,v_i) + 2\varepsilon$$

and

$$H^*(t,x,v) \leq \lim_{i\to\infty} H^*(t,x_i,v_i) + 2\varepsilon.$$

The proof follows from the arbitrariness of ε and (x,v). □

5.3 Representation of Convex Hamiltonians

This section is devoted to a representation formula for a Hamiltonian $H : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. We shall need the following assumptions

(H1) H is t-measurable and $H(t,x,\cdot)$ is convex for any $(t,x) \in [0,T] \times \mathbb{R}^n$.

(H2) For any $R > 0$
a) There exists $c_R : [0,T] \to \mathbb{R}_+$ such that for all $x, y \in B_R$ and $p \in \mathbb{R}^n$

$$|H(t,x,p) - H(t,y,p)| \leq c_R(t)(1+|p|)|x-y|.$$

b) There exists $a_R : [0,T] \to \mathbb{R}$ such that for all $x \in B_R,\ p \in \mathbb{R}^n$ and $t,\ s \in [0,T]$

$$|H(t,x,p) - H(s,x,p)| \leq (1+|p|)|a_R(t) - a_R(s)|.$$

(H3) There exists $c : [0,T] \to \mathbb{R}_+$ such that

$$|H(t,x,p) - H(t,x,q)| \leq c(t)(1+|x|)|p-q|$$

for all $(t,x) \in [0,T] \times \mathbb{R}^n$ and $p, q \in \mathbb{R}^n$.

(H4) $H^*(t,x,\cdot)$ is bounded on its domain for all $(t,x) \in [0,T] \times \mathbb{R}^n$.

(H5) For every $R > 0$ there exists $K_R : [0,T] \to \mathbb{R}_+$ such that for all $(t,x) \in [0,T] \times B_R$ and $v \in dom(H^*(t,x,\cdot))$ we have

$$H^*(t,x,v) = \max_{p \in B(0,K_R(t))} (\langle v,p\rangle - H(t,x,p)).$$

Theorem 5.3.1. *If (H1)-(H4) hold true, then there exists $f : [0,T] \times \mathbb{R}^n \times B \to \mathbb{R}^n$, measurable with respect to the first variable, such that for $l : [0,T] \times \mathbb{R}^n \times B \to \mathbb{R} \cup \{+\infty\}$ defined by $l(t,x,u) = H^*(t,x,f(t,x,u))$ we have*

(A1) $H(t,x,p) = \sup_{u\in B}(\langle p, f(t,x,u)\rangle - l(t,x,u)),\ \forall\ (t,x,p)$.

(A2) For any $R > 0$ and for all $t, s \in [0, T]$, $x, y \in B_R$, $u, v \in B$

$$\begin{cases} |f(t,x,u) - f(t,y,u)| \le 10nc_R(t)|x-y| \\ |f(t,x,u) - f(t,x,v)| \le 5n(1+R)|u-v| \\ |f(t,x,u) - f(s,x,u)| \le 10n|a_R(t) - a_R(s)|. \end{cases}$$

(A3) $|f(t,x,u)| \le c(t)(1+|x|)$ for all $(t,x,u) \in [0,T] \times \mathbb{R}^n \times B$ and f is measurable in t.

Furthermore, if (H5) is verified, then
(A4) l takes finite values and for any $R > 0$, $t, s \in [0,T]$, $x, y \in B_R$, $u, v \in B$

$$\begin{cases} |l(t,x,u) - l(s,x,u)| \le (1 + 11nK_R(t))|a_R(t) - a_R(s)| \\ |l(t,x,u) - l(t,y,u)| \le (1 + 11nK_R(t))c_R(t)|x-y|, \\ |l(t,x,u) - l(t,x,v)| \le 5nK_R(t)(1+R)|u-v|. \end{cases}$$

Define

$$F(t,x) := dom(H^*(t,x,\cdot)).$$

Lemma 5.3.2. *If (H3) holds true, then for all $(t,x) \in [0,T] \times \mathbb{R}^n$ we have $F(t,x) \subset B(0, c(t)(1+|x|))$ and $F(t,x)$ is nonempty and convex.*

Proof. Let $\omega \in F(t,x)$. By the very definition of $F(t,x)$, $\omega \in domH^*(t,x,\cdot)$. By Proposition 5.2.6, $epiH^*(t,x,\cdot)$ is convex and closed.

If there exists $p \in \partial_v H^*(t,x,\omega)$, then, by Lemma 5.2.1, $\omega \in \partial_p H(t,x,p)$. By assumption (H3), $H(t,x,\cdot)$ is $c(t)(1+|x|)$-Lipschitz. Thus $|\omega| \le c(t)(1+|x|)$. If $\partial_v H^*(t,x,\omega) = \emptyset$, then, using the separation theorem, we deduce that there exists

$$0 \ne (p, 0) \in N_{epi(H^*(t,x,\cdot))}(\omega, H^*(t,x,\omega)).$$

By Lemma 5.2.2, we can find a sequence $v_i \to \omega$ such that $(v_i, H^*(t,x,v_i)) \to (\omega, H^*(t,x,\omega))$, when $i \to \infty$ and $\partial_v H^*(t,x,v_i) \ne \emptyset$. By the first part of the proof $|v_i| \le c(t)(1+|x|)$. Consequently, $|\omega| \le c(t)(1+|x|)$. The function $H^*(t,x,\cdot)$ being convex, $F(t,x)$ is a convex subset of $\mathbb{R}^n$.

To show that for any $(t,x) \in [0,T] \times \mathbb{R}^n$, $F(t,x) \ne \emptyset$, fix any $p \in \mathbb{R}^n$. Since $\partial_p H(t,x,p) \ne \emptyset$, there exists $\omega \in \partial_p H(t,x,p)$. From Lemma 5.2.1 it follows that $H^*(t,x,\omega) \ne \infty$. □

Lemma 5.3.3. *Let $t \in [0,T]$, $x \in \mathbb{R}^n$. If $H^*(t,x,\cdot)$ is bounded on its domain, then $F(t,x)$ is closed.*

Proof. Let $v_i \to v$, $v_i \in F(t,x)$.

Since

$$\liminf_{i \to \infty} H^*(t,x,v_i) \ge H^*(t,x,v),$$

we deduce that $v \in dom(H^*(t,x,\cdot))$. Hence $F(t,x)$ is closed. □

Lemma 5.3.4. *Let (H3) hold true and $R > 0$, $t \in [0,T]$. Assume that $H^*(t,x,\cdot)$ is bounded on its domain for all $x \in B(0,R)$ and that for every $p \in \mathbb{R}^n$, $H(t,\cdot,p)$ is $c_R(t)(1+|p|)$-Lipschitz on $B(0,R)$ for some $c_R(t) \ge 0$. Then $F(t,\cdot)$ is $c_R(t)$-Lipschitz on $B(0,R)$.*

Proof. It is enough to prove that $F(t,\cdot)$ is $(c_R(t)+\varepsilon)$-Lipschitz on $B(0,R)$ for any $\varepsilon > 0$. Suppose, by contradiction, that there exist $x, y \in B(0,R)$ and $\omega \in F(t,x)$ such that $F(t,y) \bigcap B(\omega, (c_R(t)+\varepsilon)|x-y|) = \emptyset$.

By Lemmas 5.3.2, 5.3.3, $F(t,y)$ is convex and compact. Thus, by the separation theorem, for some $p \in S^{n-1}$,

$$\sup_{v \in F(t,y)} \langle p, v\rangle < \langle p, \omega\rangle - (c_R(t)+\varepsilon)|x-y|.$$

Let $\overline{v} \in F(t,y)$ be such that $\langle p, \overline{v}\rangle = \sup_{v\in F(t,y)}\langle p, v\rangle$. Then

$$p \in N_{F(t,y)}(\overline{v}) = N_{dom(H^*(t,y,\cdot))}(\overline{v})$$

and $(p,0) \in N_{epi(H^*(t,y,\cdot))}(\overline{v}, H^*(t,y,\overline{v}))$.

By Lemma 5.2.2 there exist $(p_i, q_i) \to (p, 0)$ and $v_i \to \overline{v}$ such that $q_i < 0$, $\dfrac{p_i}{|q_i|} \in \partial_v H^*(t,y,v_i)$.

Therefore

$$\left\langle \frac{p_i}{|q_i|}, v_i \right\rangle = H^*(t,y,v_i) + H\left(t, y, \frac{p_i}{|q_i|}\right).$$

Thus, for all i we have

$$\langle p_i, v_i\rangle < \langle p_i, \omega\rangle - (c_R(t)+\varepsilon)|x-y|.$$

By the Lipschitzianity assumption we have

$$H\left(t, y, \frac{p_i}{|q_i|}\right) \geq H\left(t, x, \frac{p_i}{|q_i|}\right) - c_R(t)|x-y|\left(1 + \frac{|p_i|}{|q_i|}\right).$$

Hence

$$|q_i| H^*(t,y,v_i) = \langle p_i, v_i\rangle - |q_i| H\left(t, y, \frac{p_i}{|q_i|}\right)$$

and

$$\begin{aligned}
\langle p_i, v_i\rangle - |q_i| H\left(t, y, \frac{p_i}{|q_i|}\right) &< \langle p_i, \omega\rangle - (c_R(t)+\varepsilon)|x-y| \\
&\quad -|q_i| H\left(t, x, \frac{p_i}{|q_i|}\right) + c_R(t)|x-y|(|q_i| + |p_i|) \\
&= \langle p_i, \omega\rangle - (c_R(t)+\varepsilon - c_R(t)(|q_i|+|p_i|))|x-y| - |q_i| H\left(t, x, \frac{p_i}{|q_i|}\right) \\
&= |q_i|\left(\left\langle \frac{p_i}{|q_i|}, \omega\right\rangle - H\left(t, x, \frac{p_i}{|q_i|}\right)\right) - (c_R(t)+\varepsilon - c_R(t)(|q_i|+|p_i|))|x-y|.
\end{aligned}$$

Therefore

$$|q_i| H^*(t,y,v_i) \leq |q_i| H^*(t,x,\omega) - (c_R(t)+\varepsilon - c_R(t)(|q_i|+|p_i|))|x-y|.$$

Passing to the limit and using that $H^*(t,y,\cdot)$ is bounded on its domain, it follows that $0 \leq -\varepsilon|x-y|$, leading to a contradiction. □

Lemma 5.3.5. *Let $R > 0$. Assume $(H3)$ and that $H^*(t,x,\cdot)$ is bounded on its domain for all $t \in [0,T]$ and $x \in B(0,R)$. If for a function $a_R : [0,T] \to \mathbb{R}$ and for all t, $s \in [0,T]$, $x \in B(0,R)$, $p \in \mathbb{R}^n$ we have $|H(t,x,p) - H(s,x,p)| \leq (1+|p|)|a_R(t) - a_R(s)|$, then*

$$\mathcal{H}(F(t,x), F(s,x)) \leq |a_R(t) - a_R(s)|$$

for all $t, s \in [0,T]$ and $x \in B(0,R)$.

Proof. It is enough to show that for every $\alpha > 0$, for all $t, s \in [0,T]$ and $x \in B(0,R)$,

$$\mathcal{H}(F(t,x), F(s,x)) \leq |a_R(t) - a_R(s)| + \alpha|t-s|.$$

Suppose contrary that there exist $x \in B(0,R)$, $t, s \in [0,T]$, $\omega \in F(t,x)$ such that $F(s,x) \bigcap B(\omega, |a_R(t) - a_R(s)| + \alpha|t-s|) = \emptyset$. Thus for some $p \in S^{n-1}$, $\sup_{v\in F(s,x)} \langle p, v\rangle < \langle p, \omega\rangle - |a_R(t) - a_R(s)| - \alpha|t-s|$. As $F(s,x)$ is compact, there exists $\overline{v} \in F(s,x)$ such that $\langle p, \overline{v}\rangle = \sup_{v\in F(s,x)} \langle p, v\rangle$.
Using exactly the same arguments as those in the proof of Lemma 5.3.4 we derive a contradiction. □

Consider a set valued map $J : [0,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ satisfying the following assumptions

Hypotheses (H). (i) For any $(t,x) \in [0,T] \times \mathbb{R}^n$, $J(t,x)$ is a nonempty, compact, convex subset of $\mathbb{R}^n$.
(ii) For any $x \in \mathbb{R}^n$, $J(\cdot, x)$ is measurable.
(iii) For any $R > 0$ and $t \in [0,T]$ there exists $c_R(t) \geq 0$ such that for all $x, y \in B_R$

$$\mathcal{H}(J(t,x), J(t,y)) \leq c_R(t)|x-y|.$$

Define

$$\|J(t,x)\| := \sup_{\omega\in J(t,x)} \|\omega\|.$$

Theorem 5.3.6. *Assume* (**H**). *Then there exists $f : [0,T] \times \mathbb{R}^n \times B \to \mathbb{R}^n$, measurable with respect to the first variable, such that for all $(t,x) \in [0,T] \times \mathbb{R}^n$, $f(t,x,B) = J(t,x)$ and*
i) For any $(t,u) \in [0,T] \times B$, $f(t,\cdot,u)$ is $10nc_R(t)$-Lipschitz on B_R.
ii) For any $t \in [0,T]$, $x \in \mathbb{R}^n$, $u, v \in B$

$$|f(t,x,u) - f(t,x,v)| \leq 5n\|J(t,x)\||u-v|.$$

Moreover, if for any $R > 0$ there exists a function $a_R : [0,T] \to \mathbb{R}$ such that

$$\mathcal{H}(J(t,x), J(s,x)) \leq |a_R(t) - a_R(s)| \ \ \forall\, x \in B_R,\ \forall\, t, s \in [0,T],$$

then for any $t, s \in [0,T]$, $x \in B_R$, $u \in B$

$$|f(t,x,u) - f(s,x,u)| \leq 10n|a_R(t) - a_R(s)|.$$

Theorem 5.3.7. *Let $H : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfy (H1)-(H4).*

Then there exists $f : [0,T] \times \mathbb{R}^n \times B \to \mathbb{R}^n$ as in Theorem 5.3.6 for $J(t,x) = F(t,x)$ such that for $l : [0,T] \times \mathbb{R}^n \times B \to \mathbb{R} \cup \{+\infty\}$ defined by $l(t,x,u) = H^*(t,x,f(t,x,u))$ and for any $(t,x,p) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$ we have

$$H(t,x,p) = \sup_{u\in B}(\langle p, f(t,x,u)\rangle - l(t,x,u))$$

and l is $t-$measurable.

Moreover, if (H5) is satisfied, then for every $R > 0$ and for any $t, s \in [0,T]$, $x, y \in B_R$, $u, v \in B$

$$\begin{cases} |l(t,x,u) - l(t,x,v)| \leq 5nK_R(t)(1+R)|u-v| \\ |l(t,x,u) - l(s,x,u)| \leq (1 + 11nK_R(t))|a_R(t) - a_R(s)| \\ |l(t,x,u) - l(t,y,u)| \leq (1 + 11nK_R(t))c_R(t)|x-y|. \end{cases}$$

Proof of Theorem 5.3.6. Let $P(\cdot,\cdot)$ be as in Lemma 5.2.3. For any $(t,x,u)\in[0,T]\times\mathbb{R}^n\times B$ define

$$M_x(t) := \|J(t,x)\|,\ \ \Phi(t,x,u) := P(M_x(t)u, J(t,x)),\ \ f(t,x,u) := s_n(\Phi(t,x,u)).$$

It is not difficult to check that

$$|M_x(t) - M_y(t)| \le \mathcal{H}(J(t,x), J(t,y))\ \ \forall\, x,y\in\mathbb{R}^n,\ t\in[0,T].$$

From [9, Theorem 9.7.2]) we deduce that f is measurable with respect to t, $i)$ holds true and $f(t,x,B) = J(t,x)$. From Lemma 5.2.3 and [9, Theorem 9.4.1] it follows that for any $t\in[0,T]$, $x\in\mathbb{R}^n$, $u,v\in B$

$$|f(t,x,u) - f(t,x,v)| = |s_n(\Phi(t,x,u)) - s_n(\Phi(t,x,v))| \le$$

$$n\mathcal{H}(\Phi(t,x,u),\Phi(t,x,v)) \le 5n|M_x(t)u - M_x(t)v| = 5n\|J(t,x)\||u-v|$$

and $ii)$ follows.

Now let us prove the last statement of the theorem. Fix $R>0$.
Again by [9, Theorem 9.4.1]

$$|f(t,x,u) - f(s,x,u)| = |s_n(\Phi(t,x,u)) - s_n(\Phi(s,x,u))| \le$$

$$n\mathcal{H}(\Phi(t,x,u),\Phi(s,x,u)) \le 5n(\mathcal{H}(J(t,x),J(s,x)) + |M_x(t)u - M_x(s)u|).$$

Hence for all $t,s\in[0,T]$, $x\in B_R$, $u\in B$,

$$|f(t,x,u) - f(s,x,u)| \le 5n(\mathcal{H}(J(t,x),J(s,x)) + \mathcal{H}(J(t,x),J(s,x)))$$

$$= 10n\mathcal{H}(J(t,x),J(s,x)) \le 10n|a_R(t) - a_R(s)|,$$

which completes the proof. □

Proof of Theorem 5.3.7. It is not difficult to verify, using results from [9, Chapter 8], that $F(\cdot,x)$ is measurable for all $x\in\mathbb{R}^n$. By Lemmas 5.3.2 - 5.3.5, $J(t,x) := F(t,x)$ satisfies hypothesis **(H)**. Let f be as in Theorem 5.3.6 for $J(t,x) = F(t,x)$. Then for any $(t,x,p)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n$,

$$H(t,x,p) = \sup_{v\in F(t,x)}(\langle p,v\rangle - H^*(t,x,v)) = \max_{u\in B}(\langle p, f(t,x,u)\rangle - l(t,x,u)).$$

Assume (H5) and fix $R>0$. Then for all $t\in[0,T]$, $x\in B_R$, $u\in B$

$$H^*(t,x,f(t,x,u)) = \max_{p\in B_{K_R(t)}}(\langle f(t,x,u),p\rangle - H(t,x,p))$$

Thus, by (H2)-(H3), for all $t\in[0,T]$, $x\in B_R$, $u,w\in B$,

$$|l(t,x,u) - l(t,x,w)| \le K_R(t)|f(t,x,u) - f(t,x,w)| \le 5nK_R(t)\|J(t,x)\||u-w|.$$

In the same way, for all $t,s\in[0,T]$, $x\in B_R$, $u\in B$ we can find $v\in B_{K_R}$ satisfying

$$|H^*(t,x,f(t,x,u)) - H^*(s,x,f(s,x,u))| \le K_R(t)|f(t,x,u) - f(s,x,u)| +$$

$$+|H(t,x,v) - H(s,x,v)| \le 10nK_R(t)|a_R(t) - a_R(s)| + (1+K_R(t))|a_R(t) - a_R(s)|.$$

Applying similar arguments we show that for all $t \in [0,T]$, $x, y \in B_R$, $u \in B$, there exists $v \in B_{K_R(t)}$ such that

$$|H^*(t,x,f(t,x,u)) - H^*(t,y,f(t,y,u))| \le K_R(t)|f(t,x,u) - f(t,y,u)| +$$

$$+|H(t,x,v) - H(t,y,v)| \le K_R(t)10nc_R(t)|x-y| + c_R(t)(1+K_R(t))|x-y|.$$

Therefore for any $t \in [0,T]$, $x, y \in B_R$, $u \in B$

$$|l(t,x,u) - l(t,y,u)| \le c_R(t)(1 + 11nK_R(t))|x-y|.$$

□

Proof of Theorem 5.3.1. By Lemmas 5.3.2 - 5.3.5, $J(t,x) := F(t,x)$ satisfies hypothesis **(H)**.

By Theorem 5.3.6 and Lemmas 5.3.2, 5.3.5 there exists a $t-$measurable mapping $f : [0,T] \times \mathbb{R}^n \times B \to \mathbb{R}^n$ such that for $l(t,x,u) = H^*(t,x,f(t,x,u))$, (A1) - (A3) hold true and $F(t,x) = f(t,x,B)$ for all $x \in \mathbb{R}^n$ and $t \in [0,T]$. If (H5) is satisfied, then from Theorem 5.3.7 we deduce that (A4) holds true. The proof is complete. □

5.4 Stability of Representations

In this section we show that the representation obtained in the previous section is stable with respect to Hamiltonians.

Theorem 5.4.1. *Let $H_i : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $i \ge 1$ be continuous and satisfy (H1)-(H5) with the same $c_R(\cdot), c(\cdot), a_R(\cdot)$ and $K_R(\cdot)$. Further assume that for all $R > 0$, $a_R(\cdot)$ is continuous and H_i converge uniformly on compacts to some $H : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. Consider (f_i, l_i) and (f, l) defined as in the proofs of Theorems 5.3.6 and 5.3.7 for $J(t,x) = dom(H_i^*(t,x,\cdot))$ and $J(t,x) = dom(H^*(t,x,\cdot))$ respectively. Then f_i converge to f and l_i converge to l uniformly on compacts in $[0,T] \times \mathbb{R}^n \times B$.*

To prove the above theorem we need the two lemmas below.

Lemma 5.4.2. *Under all the assumptions of Theorem 5.4.1, define*

$$F_i(t,x) = dom(H_i^*(t,x,\cdot)),\ F(t,x) = dom(H^*(t,x,\cdot)).$$

Then for every $R > 0$,

$$\lim_{i\to\infty} \sup_{t\in[0,T], x\in B_R} \mathcal{H}(F_i(t,x), F(t,x)) = 0.$$

Proof. Clearly H satisfies (H1)-(H3) with the same $c_R(\cdot), c(\cdot), a_R(\cdot)$ and $K_R(\cdot)$. Fix $t \in [0,T]$, $x \in \mathbb{R}^n$ and let $p \in Limsup_{i\to\infty} F_i(t,x)$. Thus for some $p_{i_j} \in F_{i_j}(t,x)$ we have $p_{i_j} \to p$. Fix $\varepsilon > 0$. Then for all large j and for any $v \in B_{K_R(t)}$,

$$\langle p, v\rangle - H(t,x,v) \le \langle p_{i_j}, v\rangle - H_{i_j}(t,x,v) + \varepsilon.$$

Since H_i converge to H uniformly on compacts, for some $N(t,x) \ge 0$ and for all $v \in B_{K_R(t)}$,

$$\langle p, v\rangle - H(t,x,v) \le N(t,x) + \varepsilon.$$

Therefore

$$H^*(t,x,p) \le N(t,x) + \varepsilon.$$

Hence $p \in dom(H^*(t,x,\cdot))$ and $Limsup_{i\to\infty} F_i(t,x) \subset F(t,x)$.

Furthermore, for all large i and $p \in dom(H^*(t,x,\cdot))$

$$H^*(t,x,p) = \max_{v\in B_{K_R(t)}} (\langle p,v\rangle - H(t,x,v)) \geq$$

$$\max_{v\in B_{K_R(t)}} (\langle p,v\rangle - H_i(t,x,v)) - \varepsilon = H_i^*(t,x,p) - \varepsilon.$$

Hence for any $p \in dom(H^*(t,x,\cdot))$, we have $p \in dom(H_i^*(t,x,\cdot))$, for all large i and therefore $F(t,x) \subset Liminf_{i\to\infty} F_i(t,x)$.

We have proved that for all $t \in [0,T],\ x \in \mathbb{R}^n$

$$Lim_{i\to\infty} F_i(t,x) = F(t,x). \tag{5.4.1}$$

Since for every $i \geq 1$ and for all $(t,x) \in [0,T] \times \mathbb{R}^n$, $F_i(t,x)$ is a subset of the compact set $B(0, c(t)(1+|x|))$, we deduce that

$$\lim_{i\to\infty} \mathcal{H}(F_i(t,x), F(t,x)) = 0. \tag{5.4.2}$$

We prove our lemma by contradiction, i.e. assuming that for some $\varepsilon_0 > 0$ and a subsequence i_k we have

$$\sup_{t\in[0,T], x\in B_R} \mathcal{H}(F_{i_k}(t,x), F(t,x)) > \varepsilon_0.$$

Then there exist $t_k \in [0,T], x_k \in B(0,R),\ k = 1,2,...$ such that for all k

$$\mathcal{H}(F_{i_k}(t_k,x_k), F(t_k,x_k)) > \varepsilon_0.$$

Taking a subsequence and keeping the same notations we may assume that $(t_k,x_k) \to (\overline{t},\overline{x})$, when $k \to \infty$.

By Lemmas 5.3.4 and 5.3.5

$$\mathcal{H}(F_{i_k}(t_k,x_k), F_{i_k}(\overline{t},\overline{x})) \leq c_R(\overline{t})|x_k - \overline{x}| + |a_R(t_k) - a_R(\overline{t})|.$$

$$\mathcal{H}(F(t_k,x_k), F(\overline{t},\overline{x})) \leq c_R(\overline{t})|x_k - \overline{x}| + |a_R(t_k) - a_R(\overline{t})|).$$

Therefore, by the triangular inequality, for all large k

$$\mathcal{H}(F_{i_k}(\overline{t},\overline{x}), F(\overline{t},\overline{x})) \geq \frac{\varepsilon_0}{2}.$$

This contradicts (5.4.2) and ends the proof. □

Lemma 5.4.3. *Let H_i, H be as in Theorem 5.4.1 and $F_i,\ F$ be as in Lemma 5.4.2. Define*

$$M_x^i(t) = \|F_i(t,x)\|,\ M_x(t) = \|F(t,x)\|,$$

$\Phi_i(t,x,u) = P(M_x^i(t)u, F_i(t,x))$ *and* $\Phi(t,x,u) = P(M_x(t)u, F(t,x))$. *Then,*

$$\lim_{i\to\infty} \sup_{(t,x,u)\in[0,T]\times B_R\times B} \mathcal{H}(\Phi_i(t,x,u), \Phi(t,x,u)) = 0.$$

Consequently, for all $t \in [0,T],\ x \in B_R,\ u \in B$

$$\lim_{i\to\infty} \sup_{(t,x,u)\in[0,T]\times B_R\times B} |s_n(\Phi_i(t,x,u)) - s_n(\Phi(t,x,u))| = 0.$$

Proof. If we plug in $K = F_i(t,x),\ L = F(t,x),\ z = M_x^i(t)u,\ y = M_x(t)u$ in Lemma 5.2.3, then we get

$$\mathcal{H}(\Phi_i(t,x,u),\Phi(t,x,u)) \leq 5(\mathcal{H}(F_i(t,x),F(t,x)) + |M_x^i(t) - M_x(t)||u|).$$

Thus Lemma 5.4.2 implies the first statement. Since

$$|s_n(\Phi_i(t,x,u)) - s_n(\Phi(t,x,u))| \leq n\mathcal{H}(\Phi_i(t,x,u),\Phi(t,x,u))$$

the proof is complete. □

Proof of Theorem 5.4.1. By Lemma 5.4.3, for all $(t,x,u) \in [0,T] \times \mathbb{R}^n \times B$,

$$\lim_{i\to\infty} f_i(t,x,u) = f(t,x,u).$$

Let us show that $\lim_{i\to\infty} l_i(t,x,u) = l(t,x,u)$. Fix $(t,x) \in [0,T] \times \mathbb{R}^n$ and let $R > 0$ be such that $x \in B_R$. Then for some $\overline{p} \in B_{K_R(t)}$ we have

$$l(t,x,u) = \langle \overline{p}, f(t,x,u)\rangle - H(t,x,\overline{p}).$$

Consequently

$$l_i(t,x,u) \geq \langle \overline{p}, f_i(t,x,u)\rangle - H_i(t,x,\overline{p}).$$

Passing to the limit we obtain

$$\liminf_{i\to\infty} l_i(t,x,u) \geq \langle \overline{p}, f(t,x,u)\rangle - H(t,x,\overline{p}) = l(t,x,u).$$

Consider next $\overline{p}_i \in B_{K_R(t)}$ such that

$$l_i(t,x,u) = \langle \overline{p}_i, f_i(t,x,u)\rangle - H_i(t,x,\overline{p}_i)$$

and a subsequence i_j such that

$$\lim_{j\to\infty} l_{i_j}(t,x,u) = \limsup_{i\to\infty} l_i(t,x,u).$$

Taking a subsequence and keeping the same notations without any loss of generality we may assume that $\overline{p}_{i_j} \to \widehat{p} \in B_{K_R(t)}$. Then

$$\lim_{j\to\infty} l_{i_j}(t,x,u) = \lim_{j\to\infty} (\langle \overline{p}_{i_j}, f_{i_j}(t,x,u)\rangle - H_{i_j}(t,x,\bar{p}_{i_j})) =$$

$$\langle \widehat{p}, f(t,x,u)\rangle - H(t,x,\widehat{p}) \leq H^*(t,x,f(t,x,u)).$$

Thus

$$\limsup_{i\to\infty} l_i(t,x,u) \leq l(t,x,u) \leq \liminf_{i\to\infty} l_i(t,x,u).$$

Therefore we obtain $\lim_{i\to\infty} l_i(t,x,u) = l(t,x,u)$. By Theorems 5.3.6 and 5.3.7 the families $\{f_i\}_{i\geq 1}$ and $\{l_i\}_{i\geq 1}$ are equicontinuous on compacts. Thus f_i converge to f and l_i converge to l uniformly on compacts in $[0,T] \times \mathbb{R}^n \times B$. The proof is complete. □

5.5 Case of Measurable Hamiltonians

In this section we extend results of Sections 5.3 and 5.4 to the case when the Hamiltonian is measurable in time.

Theorem 5.5.1. *Let $H : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and (H1), (H2) a), (H3), (H4) hold true.*

Then there exist $f : [0,T] \times \mathbb{R}^n \times B \to \mathbb{R}^n$, $l : [0,T] \times \mathbb{R}^n \times B \to \mathbb{R} \cup \{+\infty\}$, measurable with respect to the time variable, satisfying (A1), (A3) and such that
i) For any $(t,u) \in [0,T] \times B$, $f(t,\cdot,u)$ is $10nc_R(t)$-Lipschitz on B_R.
ii) For any $t \in [0,T]$, $x \in \mathbb{R}^n$, $u,v \in B$,

$$|f(t,x,u) - f(t,x,v)| \leq 5n\|J(t,x)\||u-v|.$$

Moreover, if (H5) is satisfied, then l takes finite values and for every $R > 0$, $x,y \in B_R$, $u,v \in B$

$$\begin{cases} |l(t,x,u) - l(t,x,v)| \leq 5nK_R(t)(1+R)|u-v| \\ |l(t,x,u) - l(t,y,u)| \leq (1 + 11nK_R(t))c_R(t)|x-y|. \end{cases}$$

The proof is similar to the one of Theorem 5.3.1 and is omitted.

Lemma 5.5.2. *Let $H_i : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $i \geq 1$ satisfy (H1), (H2) a), (H3) - (H5) with the same $c_R(\cdot), c(\cdot), K_R(\cdot)$.*

Define $\Phi_i(t,x,u)$ and $\Phi(t,x,u)$ as in Lemma 5.4.3. If for some $H : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and for almost every $t \in [0,T]$, $H_i(t,\cdot,\cdot) \to H(t,\cdot,\cdot)$ uniformly on compacts, then H is measurable with respect to time and for almost every $t \in [0,T]$,

$$\lim_{i\to\infty} \sup_{(x,u)\in B_R \times B} \mathcal{H}(\Phi_i(t,x,u), \Phi(t,x,u)) = 0.$$

Consequently for almost every $t \in [0,T]$,

$$\lim_{i\to\infty} \sup_{(x,u)\in B_R \times B} |s_n(\Phi_i(t,x,u)) - s_n(\Phi(t,x,u))| = 0.$$

Proof. The proof is similar to the one of Lemma 5.4.2 with the only difference that we keep t fixed. □

The above lemma, Theorem 5.5.1 and the proof of Theorem 5.4.1 imply the following stability of representations result.

Theorem 5.5.3. *Let $H_i : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $i \geq 1$ satisfy (H1), (H2) a), (H3)-(H5) with the same $c_R(\cdot), c(\cdot), K_R(\cdot)$. Further assume that for some $H : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and for almost all $t \in [0,T]$, $H_i(t,\cdot,\cdot) \to H(t,\cdot,\cdot)$ uniformly on compacts. Consider (f_i, l_i) and (f,l) defined as in the proofs of Theorems 5.3.6 and 5.3.7 for $J(t,x) = dom(H_i^*(t,x,\cdot))$ and $J(t,x) = dom(H^*(t,x,\cdot))$ respectively. Then for almost every $t \in [0,T]$, $f_i(t,\cdot,\cdot)$ converge to $f(t,\cdot,\cdot)$ and $l_i(t,\cdot,\cdot)$ converge to $l(t,\cdot,\cdot)$ uniformly on compacts in $\mathbb{R}^n \times B$.*

5.6 Existence of Solutions to Hamilton-Jacobi Equation

Consider $H : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and the Hamilton-Jacobi equation (5.1.1). In all the results of this section we assume that (H1), (H2) a), (H3) hold true with integrable functions $c_R(\cdot), c(\cdot)$ and that φ is lower semicontinuous and bounded from below.

We also impose the following assumption

(H6) For every $R > 0$ there exist integrable functions $\gamma_R : [0,T] \to \mathbb{R}_+$, $\psi_R : [0,T] \to \mathbb{R}$ and a constant $k_R \geq 0$ such that

$$H(t,x,v) \geq \psi_R(t) - k_R(|x| + |v|), \ \ \forall\, x \in B_R,\ v \in \mathbb{R}^n,$$

$$\sup_{v \in dom(H^*(t,x,\cdot))} |H^*(t,x,v)| \leq \gamma_R(t), \ \ \forall\, x \in B_R,\ t \in [0,T].$$

Theorem 5.6.1. *Under the above assumptions suppose that for a.e. $t \in [0,T]$, the function $H^*(t,\cdot,\cdot)$ is continuous on $\{(x, dom(H^*(t,x,\cdot))) \mid x \in \mathbb{R}^n\}$.*

Then there exists a lower semicontinuous $V : [0,T]\times\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ and a measurable set $A \subset [0,T]$ of full measure such that $V(T,\cdot) = \varphi(\cdot)$ and for all $(t,x) \in dom(V) \cap (A \times \mathbb{R}^n)$,

$$-p_t + H(t,x,-p_x) = 0, \ \ \forall\, (p_t,p_x) \in \partial_- V(t,x),$$

$$-p_t + H(t,x,-p_x) \geq 0, \ \ \forall\, (p_t,p_x) \in \partial_+ V(t,x).$$

The above theorem captures both lower semicontinuous solutions from [13, 30, 31] and viscosity solutions [22]. It has the advantage not to involve the L^1 test functions as it was done in [55].

To prove the above theorem, consider mappings $f,\ l$ as in the conclusions of Theorem 5.3.1.

Then, by the very definition of l and the lower semicontinuity of $H^*(t,\cdot,\cdot)$, (cf. Proposition 5.2.6), for all $(t,x) \in [0,T] \times \mathbb{R}^n$, the set

$$G(t,x) = \{(f(t,x,u), l(t,x,u) + r) \mid u \in B,\ r \geq 0\}$$

is convex and closed and $G(t,\cdot)$ has a closed graph.

Furthermore, it is not difficult to verify that H^* is Lebesgue-Borel-Borel measurable.

We associate to these data the Bolza optimal control problem:

$$V(t_0,x_0) = \inf\left\{\varphi(x(T)) + \int_{t_0}^{T} l(t,x(t),u(t))dt \Big|\ (x,u) \in \mathcal{S}(t_0,x_0)\right\}, \tag{5.6.1}$$

where $\mathcal{S}(t_0,x_0)$ is the set of all trajectory-control pairs of the control system

$$\left\{\begin{array}{lcll} x'(t) & = & f(t,x(t),u(t)), \ \ u(t) \in B & \quad \text{a.e.} \\ x(t_0) & = & x_0\,. & \end{array}\right. \tag{5.6.2}$$

If for some $(x,u) \in \mathcal{S}(t_0,x_0)$ the mapping $l(\cdot,x(\cdot),u(\cdot))$ is not integrable on $[t_0,T]$, then, by convention, $\int_{t_0}^T l(t,x(t),u(t))dt = +\infty$.

Observe that $V(T,\cdot) = \varphi(\cdot)$.

Proposition 5.6.2. *For all $(t_0,x_0) \in dom(V)$, the Bolza problem (5.6.1) has an optimal solution. Furthermore, V is lower semicontinuous on $[0,T] \times \mathbb{R}^n$.*

Proof. Fix $(t_0,x_0) \in dom(V)$. Then there exists $R > 0$ such that for every $x(\cdot)$ satisfying (5.6.2) we have $\|x\|_\infty \leq R$. Define

$$h(t,x,v) = \left\{\begin{array}{ll} H^*(t,x,v) & (t,x,v) \in [0,T] \times B_R \times \mathbb{R}^n \\ +\infty & \text{otherwise.} \end{array}\right.$$

From Proposition 5.2.6 we deduce that $h(t,\cdot,\cdot)$ is lower semicontinuous. Therefore, by [57], the functional

$$I(x(\cdot),v(\cdot)) = \int_{t_0}^{T} h(s,x(s),v(s))ds$$

is lower semicontinuous with respect to the L^1-strong convergence for $x(\cdot)$ and L^1-weak sequential convergence for $v(\cdot)$.

Consider a minimizing sequence of trajectory-control pairs (x_i,u_i) for the Bolza problem (5.6.1). By the Ascoli and the Alaoglu theorems there exists a subsequence $x_{i_j}(\cdot)$ converging uniformly to a Lipschitz mapping $\bar{x}(\cdot)$ such that x'_{i_j} converge weakly in L^1 to $\bar{x}'(\cdot)$.

On the other hand, for every $t \in [0,T]$ the set-valued map $B(0,R) \ni x \rightsquigarrow \Psi(t,x) := f(t,x,B)$ has closed graph contained in the compact set $B_R \times B(0,c(t)(R+1))$. Thus $\Psi(t,\cdot)$ is upper semicontinuous

Using that Ψ has closed convex nonempty values and the same proof as the one of [9, Theorem 7.2.2] we deduce that for almost every $t \in [0,T]$, $\bar{x}'(t) \in \Psi(t,\bar{x}(t))$. By the measurable selection theorem there exists a measurable control $\bar{u} : [t_0,T] \to B$ satisfying $\bar{x}'(\cdot) = f(\cdot,\bar{x}(\cdot),\bar{u}(\cdot))$ a.e. in $[t_0,T]$. Since

$$I(x_{i_j}(\cdot),x'_{i_j}(\cdot)) = \int_{t_0}^{T} h(s,x_{i_j}(s),x'_{i_j}(s))ds = \int_{t_0}^{T} l(s,x_{i_j}(s),u_{i_j}(s))ds,$$

we obtain

$$\liminf_{j\to\infty} \int_{t_0}^{T} l(s,x_{i_j}(s),u_{i_j}(s))ds \geq I(\bar{x}(\cdot),\bar{x}'(\cdot)) = \int_{t_0}^{T} l(s,\bar{x}(s),\bar{u}(s))ds.$$

On the other hand,

$$\liminf_{j\to\infty} \varphi(x_{i_j}(T)) \geq \varphi(\bar{x}(T)).$$

Hence $(\bar{x}(\cdot),\bar{u}(\cdot))$ is optimal.

To show the lower semicontinuity of V, pick a sequence $(t_i,y_i) \in dom(V)$ converging to some $(t_0,x_0) \in [0,T] \times \mathbb{R}^n$. For every i consider an optimal trajectory-control pair $(x_i(\cdot),u_i(\cdot))$ of the Bolza problem (5.6.1) with (t_0,x_0) replaced by (t_i,y_i). We extend $(x_i(\cdot),u_i(\cdot))$ on the time interval $[0,T]$ as trajectories of (5.6.2) with (t_0,x_0) replaced by (t_i,y_i).

In the same way as before we extract a subsequence $x_{i_j}(\cdot)$ converging uniformly to a mapping $\bar{x}(\cdot)$ such that $x'_{i_j}(\cdot)$ converge weakly in L^1 to $\bar{x}'(\cdot)$. Consider a measurable control $\bar{u} : [t_0,T] \to B$ satisfying $\bar{x}'(\cdot) = f(\cdot,\bar{x}(\cdot),\bar{u}(\cdot))$ a.e. in $[t_0,T]$. Using again [57] we deduce that for any $0 < \varepsilon < 1 - t_0$

$$\liminf_{i\to\infty} \int_{t_0+\varepsilon}^{T} h(s,x_{i_j}(s),x'_{i_j}(s))ds \geq \int_{t_0+\varepsilon}^{T} h(s,\bar{x}(s),\bar{x}'(s))ds$$

$$= \int_{t_0+\varepsilon}^{T} l(s,\bar{x}(s),\bar{u}(s))ds.$$

The arbitrariness of $\varepsilon > 0$ and the second inequality in the assumption (H6) imply that

$$\liminf_{i\to\infty} \int_{t_i}^{T} l(s,x_{i_j}(s),u_{i_j}(s))ds \geq \int_{t_0}^{T} l(s,\bar{x}(s),\bar{u}(s))ds.$$

This and the lower semicontinuity of φ end the proof. □

Lemma 5.6.3. *There exists a measurable set $A \subset (0,T)$ of full measure in $(0,T)$ such that for all $(t_0, x_0) \in dom(V) \cap (A \times \mathbb{R}^n)$ we can find $u_0 \in B$ satisfying*

$$D_\uparrow V(t_0,x_0)(1, f(t_0,x_0,u_0)) \leq -l(t_0,x_0,u_0).$$

In particular, V is a viscosity supersolution of (5.1.1): for any $(p_t, p_x) \in \partial_- V(t_0,x_0)$ it holds $-p_t + H(t,x,-p_x) \geq 0$.

Proof. Observe that for every optimal trajectory control pair $(\bar{x}, \bar{u})$ of (5.6.1) and the absolutely continuous function $y : [t_0, T] \to \mathbb{R}$ defined by

$$y(t) = \int_{t_0}^t l(s, \bar{x}(s), \bar{u}(s))ds$$

we have

$$(\bar{x}, y)'(t) \in G(t, \bar{x}(t)) \text{ a.e. in } [t_0, T].$$

For every integer $i \geq 1$ define the set valued map $\Psi_i : [0,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^{n+1}$ by

$$\Psi_i(t,x) = \begin{cases} G(t,x) \cap (\mathbb{R}^n \times B(0, \gamma_{2i}(t))) & |x| < 2i \\ c(t)(1+2i)B \times B(0,\gamma_{2i}(t)) & |x| \geq 2i \end{cases}$$

It is measurable with respect to t. Moreover $\Psi_i(t, \cdot)$ has convex compact nonempty images and closed graph.

Observe that for any $r > 0$ and for all $x_0 \in B_r$ and $t_0 \in [0,T]$ there exists $i \geq 1$ such that any trajectory of control system (5.6.2) satisfies $x([0,T]) \subset B(0,i)$.

From [39, Corollary 2.7] it follows that there exists a set $A_i \subset [0,T]$ of full measure such that for all $(t_0, x_0) \in A_i \times \mathbb{R}^n$ and any optimal trajectory control pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ of (5.6.1) satisfying $x([t_0,T]) \subset B(0,i)$ we have

$$\emptyset \neq Limsup_{h \to 0+} \frac{(\bar{x}(t_0+h), \bar{y}(t_0+h)) - (x_0, 0)}{h} \subset G(t_0, x_0), \tag{5.6.3}$$

where $\bar{y} : [t_0, T] \to \mathbb{R}$ is defined by

$$\bar{y}(t) = \int_{t_0}^t l(s, \bar{x}(s), \bar{u}(s))ds.$$

Set $A = \cap_{i=1}^\infty A_i$. Then for all $(t_0,x_0) \in A \times \mathbb{R}^n$ and any optimal trajectory control pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ of (5.6.1) the relation (5.6.3) holds true.

Fix $(t_0,x_0) \in A \times \mathbb{R}^n$ and let $u_0 \in U,\ r_0 \geq 0$ be such that

$$(f(t_0,x_0,u_0), l(t_0,x_0,u_0) + r_0) \in Limsup_{h \to 0+} \frac{(\bar{x}(t_0+h), \bar{y}(t_0+h)) - (\bar{x}_0, 0)}{h}.$$

By the dynamic programming principle

$$V(t_0+h, \bar{x}(t_0+h)) + \int_{t_0}^{t_0+h} l(s, \bar{x}(s), \bar{u}(s))ds = V(t_0,x_0)$$

Consequently,

$$D_\uparrow V(t_0,x_0)(1, f(t_0,x_0,u_0)) \leq -l(t_0,x_0,u_0).$$

□

Lemma 5.6.4. *Assume in addition that for a.e.* $t \in [0,T]$*, the function* $H^*(t,\cdot,\cdot)$ *is continuous on the set* $\{(x, dom(H^*(t,x,\cdot))) \mid x \in \mathbb{R}^n\}$.

Then there exists a measurable subset $A \subset (0,T)$ *of full measure in* $(0,T)$ *such that for all* $(t_0,x_0) \in dom(V) \cap (A \times \mathbb{R}^n)$ *and any* $u \in B$ *we have*

$$D_{\uparrow}V(t_0,x_0)(-1,-f(t_0,x_0,u)) \leq l(t_0,x_0,u)$$

and

$$D_{\downarrow}V(t_0,x_0)(1,f(t_0,x_0,u)) \geq -l(t_0,x_0,u).$$

Thus

$$-p_t + H(t,x,-p_x) \leq 0,\ \forall\ (p_t,p_x) \in \partial_- V(t_0,x_0)$$

and

$$-p_t + H(t,x,-p_x) \leq 0,\ \forall\ (p_t,p_x) \in \partial_+ V(t_0,x_0).$$

In particular, V *is a viscosity subsolution of (5.1.1).*

Proof. We first observe that the function $\mathbb{R}^n \times B \ni (x,u) \to H^*(t,x,f(t,x,u))$ is continuous. For all $i \geq 1,\ t \in [0,T]$ define

$$\psi_i(t,x,u) = \begin{cases} (f(t,x,u), l(t,x,u)) & |x| \leq 2i \\ (f(t,\pi_i x,u), l(t,\pi_i x,u)) & |x| > 2i, \end{cases}$$

where π_i denotes the projection of x on $B(0,2i)$.

From [39, Corollary 2.8] it follows that for a subset $A_i \subset [0,T]$ of full measure and for all $t_0 \in A_i,\ x_0 \in B(0,i),\ u \in B$ the solution $x(\cdot)$ of (5.6.2) corresponding to the constant control u satisfies $x'(t_0) = f(t_0,x_0,u)$ and

$$\lim_{h\to 0+} \frac{1}{h} \int_{t_0}^{t_0+h} l(s,x(s),u)ds = l(t_0,x_0,u),$$

$$\lim_{h\to 0+} \frac{1}{h} \int_{t_0-h}^{t_0} l(s,x(s),u)ds = -l(t_0,x_0,u).$$

Set $A = \cap_{i=1}^{\infty} A_i$. Then for all $(t_0,x_0) \in A \times \mathbb{R}^n$ and any $u \in B$ the solution $x(\cdot)$ of (5.6.2) corresponding to the constant control u is as above. By the dynamic programming principle,

$$V(t_0+h,\ x(t_0+h)) + \int_{t_0}^{t_0+h} l(s,x(s),u)ds \geq V(t_0,x_0)$$

and

$$V(t_0-h, x(t_0-h)) \leq V(t_0,x_0) + \int_{t_0-h}^{t_0} l(s,x(s),u)ds.$$

Consequently, for any $u \in B$,

$$D_{\uparrow}V(t_0,x_0)(-1,-f(t_0,x_0,u)) \leq l(t_0,x_0,u)$$

and

$$D_{\downarrow}V(t_0,x_0)(1,f(t_0,x_0,u)) \geq -l(t_0,x_0,u).$$

□

We have proved the existence Theorem 5.6.1. Moreover the considered solution of (5.1.1) is represented by the value function of a Bolza optimal control problem.

Uniqueness of lower semicontinuous solutions can be investigated in the same way as in [39] by applying viability and invariance theorems from [39], [36] on the epigraph of a solution. This is however beyond the scope of the present paper.

Acknowledgements. This work was co-funded by the European Union under the 7th Framework Programme "FP7-PEOPLE-2010-ITN", grant agreement number 264735-SADCO.

Chapter 6

Stability of solutions to Hamilton-Jacobi equations under state constraints

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Submitted for publication.

Abstract. In the present paper we investigate stability of solutions of Hamilton-Jacobi-Bellman equations under state constraints by studying stability of value functions of a suitable family of Bolza optimal control problems under state constraints. The stability is guaranteed by the classical assumptions imposed on Hamiltonians and an inward pointing condition on state constraints.

Keywords: Hamilton-Jacobi equation, optimal control, Bolza problem, viscosity solution, state constraints, stability of solutions.

AMS Mathematics Subject Classification 2010: 49L25, 26E25, 34A60.

6.1 Introduction

Consider the following state constrained Hamilton-Jacobi-Bellman equation

$$\begin{cases} -v_t(t,x) + H(t,x,-v_x(t,x)) = 0, \ (t,x) \in [0,T] \times K \\ v(T,x) = \varphi(x), \end{cases}$$

where $T > 0$, K is a given nonempty and closed subset of $\mathbb{R}^n$, the Hamiltonian $H : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is convex in the last variable and $\varphi : \mathbb{R}^n \to \mathbb{R}$.

In [40] it was proved that the Hamiltonian H can be represented by mappings $f : [0,T] \times \mathbb{R}^n \times B \to \mathbb{R}^n$ and $l : [0,T] \times \mathbb{R}^n \times B \to \mathbb{R}$, which inherit Lipschitz type regularity

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properties of H and such that $H(t,x,p) = \max_{u \in B}(\langle p, f(t,x,u)\rangle - l(t,x,u))$, where B is the closed unit ball in $\mathbb{R}^n$.

We can associate to f and l the following Bolza optimal control problem under state constraints

$$minimize\{\varphi(x(T)) + \int_0^T l(t,x(t),u(t))\,\mathrm{d}t\} \tag{6.1.1}$$

over all trajectory-control pairs $(x,u)(\cdot)$ of the control system

$$\dot{x}(t) = f(t,x(t),u(t)),\ u(t) \in B \tag{6.1.2}$$

satisfying the initial condition

$$x(0) = x_0,$$

and state constraints

$$x(t) \in K\ for\ all\ t \in [0,T].$$

The value function associated to the Bolza optimal control problem (6.1.1)-(6.1.2) is defined by: for all $t_0 \in [0,T]$ and $y_0 \in \mathbb{R}^n$,

$$V(t_0,y_0) = \inf\{\varphi(x(T)) + \int_{t_0}^T l(t,x(t),u(t))\,\mathrm{d}t : (x,u) \in S(t_0,y_0)\},$$

where $S(t_0,y_0)$ denotes the set of all trajectory control pairs of the control system (6.1.2) satisfying the initial condition $x(t_0) = y_0$ and state constraints $x(t) \in K$, for all $t \in [0,T]$.

It is well known that in the absence of state constraints the value function of the Bolza problem satisfies the Hamilton-Jacobi-Bellman equation in a generalized sense. Namely, under some technical assumptions, V is a unique viscosity solution of the Hamilton-Jacobi equation, whenever $K = \mathbb{R}^n$, see Crandall-Lions [23] for the definition of viscosity solution.

For the case with no state constraints there is large literature, where under appropriate assumptions it is proved that V is the unique viscosity solution of Hamilton-Jacobi-Bellman equation, cf. [18], [31].

Several papers were devoted to Hamilton-Jacobi-Bellman equations under state constraints, cf. [19], [41]. The uniqueness of solution of Hamilton-Jacobi-Bellman equation was proved by different authors under the hypotheses which include the inward-pointing condition (IPC). Soner [70] has considered inward pointing condition for K having a smooth boundary and investigated the infinite horizon optimal control problem. Capuzzo-Dolcetta and Lions [19] have shown when $K = \bar{\Omega}$, where Ω is an open set with sufficiently smooth boundary and satisfies (IPC) that the value function is continuous and is the unique viscosity solution to Hamilton-Jacobi-Bellman equation. The inward pointing condition is an important property in investigation of uniqueness of solutions to Hamilton-Jacobi-Bellman equation under state constraints, because it allows to approximate (in the sense of uniform convergence) feasible trajectories by trajectories staying in the interior of the set K, see for example [37] and [17], [34] for the most recent neighboring feasible trajectories (NFT) theorems concerning such approximations. In order to investigate the discontinuous solutions to Hamilton-Jacobi-Bellman equation, Ishii and Koike have expressed in [51] the inward pointing condition using "inward" trajectories of a control system, which is not simple to verify.

In general, the value function of the Bolza optimal control problem may be not continuous (even if all data are smooth). In [37] and [35] Frankowska and Plaskacz have proved the uniqueness results for Hamilton-Jacobi-Bellman equation by extending the inward pointing condition to constraints having nonsmooth boundary and an empty interior using an interplay between the sets $f(t,x,U)$ and tangents to the set K instead of trajectories, such approach is more convenient, because to check the conditions of [51] one has to check their (IPC) on trajectories of control system, while in [35] the verification involves only the data f and K.

In the present paper we investigate stability of solutions of Hamilton-Jacobi-Bellman equations by investigating stability of value functions of Bolza problems. The stability is guaranteed by the classical assumptions imposed on Hamiltonians and an inward pointing condition on state constraints. We show that under appropriate assumptions, V is a unique viscosity solution to Hamilton-Jacobi-Bellman equation. This allows to conclude that solutions are stable with respect to Hamiltonians and state constraints.

The outline of the paper is as follows: In Section 6.2 we recall some notions and introduce some notations. In Section 6.3 we investigate the stability of value functions of Bolza problems. In Section 6.4 we associate with a Hamilton-Jacobi-Bellman equation (with the Hamiltonian convex in the last variable) a Bolza optimal control problem. In Section 6.5 we prove the uniqueness of solutions of Hamilton-Jacobi-Bellman equation and their continuous dependence on data.

6.2 Preliminaries and Notations

The notation $B(x_0,R)$ stands for the closed ball in $\mathbb{R}^n$ of center $x_0 \in \mathbb{R}^n$ and radius $R \geq 0$ and $RB := B(0,R)$, $B := B(0,1)$. We denote by $\langle p,v\rangle$ the scalar product of p, $v \in \mathbb{R}^n$ and by $|x|$ the Euclidean norm. For a bounded function $f:\Omega \to \mathbb{R}$ we define $\|f\|_\infty = \sup\{\,|f(x)| : x \in \Omega\,\}$. For a set $X \subset \mathbb{R}^n$, denote by coX its convex hull. For an extended real-valued function $f:\mathbb{R}^n \to \mathbb{R}\cup\{\pm\infty\}$, $f\,|_K$ stands for the restriction of f to K.

Let A be a metric space with the distance d and X be a subset of A. The distance from $x \in A$ to X is defined by

$$d(x,X) := \inf_{y\in X} d(x,y),$$

where we have set $d(x,\emptyset) = +\infty$. We denote by ∂X the boundary of X.

Let $\{X_i\}_{i\geq 1}$ be a family of subsets of A. The subset

$$Limsup_{i\to\infty} X_i := \{x \in A : \liminf_{i\to\infty} d(x,X_i) = 0\} =$$

$$= \{x \in A :\ for\ every\ open\ neighbourhood\ U\ of\ x,\ U \cap X_i \neq \emptyset\ for\ infinitely\ many\ i\},$$

is called the upper limit of the sequence X_i and the subset

$$Liminf_{i\to\infty} X_i := \{x \in A : \limsup_{i\to\infty} d(x,X_i) = 0\} =$$

$$= \{x \in A :\ for\ every\ open\ neighbourhood\ U\ of\ x,\ U \cap X_i \neq \emptyset\ for\ all\ large\ enough\ i\},$$

is called its lower limit. A subset X is said to be the (Kuratowski) set limit of the sequence X_i if

$$X = Liminf_{i\to\infty} X_i = Limsup_{i\to\infty} X_i =: Lim_{i\to\infty} X_i.$$

For arbitrary subsets X,Y of $\mathbb{R}^n$, the extended Hausdorff distance between X and Y is defined by

$$\mathcal{H}aus(X,Y) := max\{\sup_{x\in X} d(x,Y), \sup_{x\in Y} d(x,X)\} \in \mathbb{R}\cup\{+\infty\},$$

which may be equal to $+\infty$ when X or Y is unbounded or empty.

It is well known that if X_i are subsets of a given compact set, then

$$X = Lim_{i\to\infty} X_i \Leftrightarrow \lim_{i\to\infty} \mathcal{H}aus(X_i, X) = 0.$$

Let $T > 0$, $F(\cdot,\cdot) : [0,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ be a multifunction with compact, non-empty values. Consider $t_0 \in [0,T)$ and the following differential inclusion

$$\dot{x}(t) \in F(t, x(t)),\ a.e.\ t \in [t_0, T]. \tag{6.2.1}$$

Solutions to differential inclusion (6.2.1) are understood in the Caratheodory sense, i.e. absolutely continuous functions verifying (6.2.1) almost everywhere. We denote by $\bar{S}_{[t_0,T]}(x_0)$ the set of absolutely continuous solutions $x(\cdot)$ of (6.2.1) defined on $[t_0,T]$ and satisfying the initial condition $x(t_0) = x_0$.

Let $K \subset \mathbb{R}^n$ be a closed, non-empty set. Consider the following state constrained differential inclusion

$$\begin{cases} \dot{x}(t) \in F(t, x(t)),\ a.e.\ t \in [t_0, T] \\ x(t) \in K,\ \forall t \in [t_0, T]. \end{cases} \tag{6.2.2}$$

The very proof of Theorem 2.3 from [17] implies the following result (the so-called neighboring feasible trajectories theorem) stated in a slightly different way than [17, Theorem 2.3].

Theorem 6.2.1 (NFT)**.** *Let $r_0 > 0$. Assume that for some constant $c > 0$ and for $R = e^{cT}(r_0 + 1)$ the following hypotheses hold true*

i). $\max_{v\in F(t,x)} |v| \le c(1 + |x|)$, *for any $x \in \mathbb{R}^n$ and for $t \in [0,T]$.*

ii). There exists $c_R(\cdot) \in L^1(0,T)$ such that for all $x, x' \in RB$ and a.e. $t \in [0,T]$

$$F(t, x') \subset F(t, x) + c_R(t)|x - x'|B,$$

*iii). **(IPC)** There exist $\varepsilon > 0, \eta > 0$ such that for any $(t,x) \in [0,T] \times (\partial K + \eta B) \cap RB \cap K$ we can find $v \in coF(t,x)$ satisfying $x' + [0,\varepsilon](v + \varepsilon B) \subset K$, for all $x' \in (x + \varepsilon B) \cap K$.*

iv). For an absolutely continuous function $a_R : [0,T] \to \mathbb{R}$ and for any $x \in K \cap RB$ and $0 \le s < t \le T$

$$F(s, x) \subset F(t, x) + \int_s^t a_R(\tau) d\tau B.$$

Then there exists $C > 0$ depending only on $\varepsilon, \eta, c, c_R(\cdot)$ and $a_R(\cdot)$ such that for any $t_0 \in [0,T)$ and any solution $\hat{x}(\cdot)$ of (6.2.1), with $\hat{x}(t_0) \in K \cap (e^{ct_0}(r_0+1) - 1)B$, we can find a solution $x(\cdot)$ of (6.2.2) satisfying $x(t_0) = \hat{x}(t_0)$, $x(t) \in IntK$ for all $t \in (t_0, T]$ and

$$|\hat{x}(\cdot) - x(\cdot)|_{C([t_0,T],\mathbb{R}^n)} \le C \max_{t\in[t_0,T]} dist(\hat{x}(t), K).$$

Definition 6.2.2. *Let $i \ge 1$ and $K_i \subset \mathbb{R}^n$ be closed, non-empty sets. For $T > 0$ consider $V_i : [0,T] \times K_i \to \mathbb{R}$. We say that V_i are equicontinuous uniformly in i, if for any $\varepsilon > 0$, there exists $\delta > 0$, such that for any i and any $x, y \in K_i$, $t, s \in [0,T]$ with $|x-y| + |t-s| \le \delta$*

$$|V_i(t,x) - V_i(s,y)| \le \varepsilon.$$

Definition 6.2.3. *Let $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$.*

i) $Dom(\phi)$ is the set of all $x_0 \in \mathbb{R}^n$, such that $\phi(x_0) \neq \pm\infty$.

ii) The epigraph of ϕ is defined by

$$epi(\phi) = \{(x, a) : x \in \mathbb{R}^n, a \in \mathbb{R}, a \geq \phi(x)\}.$$

The hypograph of ϕ is defined by

$$hyp(\phi) = \{(x, a) : x \in \mathbb{R}^n, a \in \mathbb{R}, a \leq \phi(x)\}.$$

iii) The subdifferential of ϕ at $x_0 \in Dom(\phi)$ is defined by

$$\partial_-\phi(x_0) = \{p \in \mathbb{R}^n : \liminf_{x \to x_0} \frac{\phi(x) - \phi(x_0) - \langle p, x - x_0 \rangle}{|x - x_0|} \geq 0\}.$$

The superdifferential of ϕ at $x_0 \in Dom(\phi)$ is defined by

$$\partial_+\phi(x_0) = \{p \in \mathbb{R}^n : \limsup_{x \to x_0} \frac{\phi(x) - \phi(x_0) - \langle p, x - x_0 \rangle}{|x - x_0|} \leq 0\}.$$

iv) The contingent epiderivative of ϕ at $x_0 \in Dom(\phi)$ in the direction $u \in \mathbb{R}^n$ is defined by

$$D_\uparrow\phi(x_0)(u) = \liminf_{h \to 0+, v \to u} \frac{\phi(x_0 + hv) - \phi(x_0)}{h}.$$

The contingent hypoderivative of ϕ at $x_0 \in Dom(\phi)$ in the direction $u \in \mathbb{R}^n$ is defined by

$$D_\downarrow\phi(x_0)(u) = \limsup_{h \to 0+, v \to u} \frac{\phi(x_0 + hv) - \phi(x_0)}{h}.$$

Definition 6.2.4. *Let $X \subset \mathbb{R}^n$. We call $d \in \mathbb{R}^n$ a tangent direction (in the sense of Bouligand) to X at point $x \in X$, if there exist sequences $\{x_k\}_{k \in \mathbb{N}}$ and $\{t_k\}_{k \in \mathbb{N}}$ such that*

$$\{x_k\} \subset X, \quad t_k \downarrow 0, \quad \frac{x_k - x}{t_k} \to d.$$

The set of tangent directions to X at x is called a tangent cone (in the sense of Bouligand) for X at x and is denoted by $T_X(x)$.

Definition 6.2.5. *We define a normal cone (in the sense of Bouligand) to a set $X \subset \mathbb{R}^n$ at point $x \in X$ by*

$$N_X(x) \doteq \{p \in \mathbb{R}^n : \langle p, v \rangle \leq 0, \forall v \in T_X(x)\}.$$

Lemma 6.2.6. *Let K be a closed set in $\mathbb{R}^n$ and $F : K \rightsquigarrow \mathbb{R}^n$ be lower semicontinuous with closed images. Then the following are equivalent*

i) $F(x) \subset T_K(x)$, for $\forall x \in K$.

ii) $F(x) \subset \bar{co}T_K(x)$, for $\forall x \in K$.

Proof. i) $\Rightarrow$ ii), is immediate.

ii) $\Rightarrow$ i), from [9, Theorem 4.1.10] we deduce that

$$Liminf_{x \to_K x_0} F(x) \subset Liminf_{x \to_K x_0} \bar{co}T_K(x) \subset T_K(x_0),$$

where $\to_K$ denotes the convergence in the set K.

As F is lower semicontinuous, $F(x_0) \subset Liminf_{x \to x_0} F(x)$.

Which ends the proof. □

Lemma 6.2.7 ([31])**.** *Let $\phi : \mathbb{R}^n \to \mathbb{R}$, $x_0 \in \mathbb{R}^n$. Then $p \in \partial_-\phi(x_0)$ if and only if for any $v \in \mathbb{R}^n$*

$$D_\uparrow\phi(x_0)(v) \geq \langle p, v \rangle$$

and $p \in \partial_+\phi(x_0)$ if and only if for any $u \in \mathbb{R}^n$

$$D_\downarrow\phi(x_0)(u) \leq \langle p, u \rangle.$$

6.3 Stability of value functions of Bolza problems

Let $T > 0$, U be a compact metric space, K and K_i be nonempty, closed subsets of $\mathbb{R}^n$ for $i = 1, 2, ...$, controls $u(\cdot)$ be Lebesgue measurable maps on $[0, T]$ taking values in U, $y_0 \in \mathbb{R}^n$ and $\varphi : \mathbb{R}^n \to \mathbb{R}$, $\varphi_i : \mathbb{R}^n \to \mathbb{R}$ be continuous. Consider continuous functions $f : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$, $f_i : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$, $l : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$, $l_i : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$, $i = 1, 2, ...$ and the following Bolza optimal control problems:

$$(P) \begin{cases} \min \int_0^T l(s, x(s), u(s))\, \mathrm{d}s + \varphi(x(T)) \\ \dot{x}(s) = f(s, x(s), u(s)),\ u(s) \in U \ a.e.\ in\ [0, T] \\ x(0) = y_0 \\ x(s) \in K,\ \forall s \in [0, T]. \end{cases} \tag{6.3.1}$$

$$(P_i) \begin{cases} \min \int_0^T l_i(s, x(s), u(s))\, \mathrm{d}s + \varphi_i(x(T)) \\ \dot{x}(s) = f_i(s, x(s), u(s)),\ u(s) \in U \ a.e.\ in\ [0, T] \\ x(0) = y_0 \\ x(s) \in K_i,\ \forall s \in [0, T]. \end{cases} \tag{6.3.2}$$

We impose the following assumptions on f and l.

(A1) For any $R > 0$ there exist an integrable function $c_R : [0, T] \to \mathbb{R}_+$ and an absolutely continuous function $a_R : [0, T] \to \mathbb{R}$ such that for all $t, s \in [0, T]$, $x, y \in RB$, $u \in U$

$$\begin{cases} |f(t, x, u) - f(t, y, u)| + |l(t, x, u) - l(t, y, u)| \leq c_R(t)|x - y| \\ |f(t, x, u) - f(s, x, u)| + |l(t, x, u) - l(s, x, u)| \leq |a_R(t) - a_R(s)|. \end{cases}$$

(A2) There exists $c > 0$ such that $|f(t, x, u)| \leq c(1+|x|)$ for all $(t, x, u) \in [0, T] \times \mathbb{R}^n \times U$.

For any $(t_0, y_0) \in [0, T] \times \mathbb{R}^n$ denote by $S_{[t_0,T]}(y_0)$ the set of all trajectory-control pairs of the control system under state constraint

$$\begin{cases} \dot{x}(s) = f(s, x(s), u(s)),\ u(s) \in U \ a.e.\ in\ [t_0, T] \\ x(t_0) = y_0 \\ x(s) \in K,\ \forall s \in [t_0, T] \end{cases} \tag{6.3.3}$$

and by $S^i_{[t_0,T]}(y_0)$ the set of all trajectory-control pairs of the following control system under state constraint

$$\begin{cases} \dot{x}(s) = f_i(s, x(s), u(s)),\ u(s) \in U \ a.e.\ in\ [t_0, T] \\ x(t_0) = y_0 \\ x(s) \in K_i,\ \forall s \in [t_0, T]. \end{cases} \tag{6.3.4}$$

The value function of the Bolza optimal control problem (P) is defined by: $\forall (t_0, y_0) \in [0, T] \times \mathbb{R}^n$

$$V(t_0, y_0) = \inf\{\int_{t_0}^T l(s, x(s), u(s))\, \mathrm{d}s + \varphi(x(T)) : (x, u) \in S_{[t_0,T]}(y_0)\}. \tag{6.3.5}$$

Similarly, the value function of the Bolza optimal control problem (P_i) is defined by: $\forall (t_0, y_0) \in [0, T] \times \mathbb{R}^n$

$$V_i(t_0, y_0) = \inf\{\int_{t_0}^T l_i(s, x(s), u(s))\, \mathrm{d}s + \varphi_i(x(T)) : (x, u) \in S^i_{[t_0,T]}(y_0)\}. \tag{6.3.6}$$

In the above we set $V(t_0, y_0) = +\infty$, if $S_{[t_0,T]}(y_0) = \emptyset$, respectively $V_i(t_0, y_0) = +\infty$, if $S^i_{[t_0,T]}(y_0) = \emptyset$.

We assume that the closed sets K and K_i are defined by the multiple inequality constraints, namely let $g_i^j : \mathbb{R}^n \to \mathbb{R}$ and $g^j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, .., m$, $i = 1, 2, ...$ be given continuously differentiable functions satisfying

(A3) Regularity.

i) For any $R > 0$ there exists $A_R > 0$ such that $|\nabla g_i^j(x)| \leq A_R$, for any $x \in RB$ and ∇g_i^j is A_R-Lipschitz on RB, $i = 1, 2, ...$, $j = 1, 2, ..., m$.

ii) $\nabla g_i^j \to \nabla g^j$ uniformly on compacts and $g_i^j(0) \to g^j(0)$, when $i \to \infty$, for any $j = 1, ..., m$.

Consider closed sets

$$K_i \doteq \bigcap_{j=1}^{m} \{x : g_i^j(x) \leq 0\} \tag{6.3.7}$$

$$K \doteq \bigcap_{j=1}^{m} \{x : g^j(x) \leq 0\}. \tag{6.3.8}$$

For any $x \in \mathbb{R}^n$ denote by $I(x)$ the set of active indices at x for $g(\cdot) = (g^1(\cdot), ..., g^m(\cdot))$, i.e.

$$I(x) = \{j : g^j(x) = 0\}.$$

(A4) Inward pointing condition.

For any $R > 0$ there exists $\rho_R > 0$ such that for every $x \in K \cap RB$ with $I(x) \neq \emptyset$ and every $s \in [0, T]$

$$\inf_{v \in cof(s,x,U)} \max_{j \in I(x)} \langle \nabla g^j(x), v \rangle \leq -\rho_R.$$

Lemma 6.3.1. *Let $K, K_i \subset \mathbb{R}^n$ defined above be non-empty and (A2), (A3), (A4) hold true. If f_i converge to f uniformly on compacts, then for every $R > 0$ there exist $\eta_R > 0$, $\varepsilon > 0$, $i_0 \geq 1$ such that for all $i \geq i_0$, $t \in [0, T]$ and $x \in (\partial K_i + \eta_R B) \cap RB \cap K_i$ we can find $v_{x,t} \in cof_i(t, x, U)$ satisfying $x' + [0, \varepsilon](v_{x,t} + \varepsilon B) \subset K_i$, for all $x' \in (x + \varepsilon B) \cap K_i$.*

Proof. Fix $R > 0$. We claim that there exist $i_0 \geq 0$, $0 < \eta'_R < 1$ such that for all $s \in [0, T]$, $x \in RB \cap K_i$, $j = 1, ..., m$, $i \geq i_0$ satisfying $-\eta'_R \leq g_i^j(x) \leq 0$, we have

$$\langle \nabla g_i^j(x), v_{x,s} \rangle \leq -\frac{\rho_R}{2},$$

for some $v_{x,s} \in cof_i(s, x, U)$. The existence of such η'_R and i_0 can be proved by a contradiction argument. Indeed, assume that for some $\eta'_k \to 0$ and $i_k \to \infty$, when $k \to \infty$, there exist $s_k \in [0, T]$, $x_k \in RB \cap K_{i_k}$ and $j \in \{1, ..., m\}$ satisfying $-\eta'_k \leq g_{i_k}^j(\cdot) \leq 0$ and

$$\langle \nabla g_{i_k}^j(x), v \rangle > -\frac{\rho_R}{2},$$

for any $v \in cof_{i_k}(s_k, x_k, U)$. Taking a subsequence, we may assume that $x_k \to x$ and $s_k \to s$, as $k \to \infty$. Then, as f_i converge to f uniformly on compacts, using (A3)-ii) and passing to the limit when $k \to \infty$ we deduce that $g^j(x) = 0$ and

$$\langle \nabla g^j(x), v \rangle \geq -\frac{\rho_R}{2}, \ \forall v \in cof(s, x, U).$$

Then

$$\inf_{v\in cof(s,x,U)} \max_{j\in I(x)} \langle \nabla g^j(x), v\rangle \geq -\frac{\rho_R}{2}$$

contrary to our assumption.

Consider

$$0 < \varepsilon < \min\{\frac{\rho_R}{8A_{R+1}(c(1+R)+1)^2}, \frac{\eta'_R}{(A_{R+1}+1)(c(1+R)+1)}, 1\}.$$

Fix $i \geq i_0$, $x \in K_i \cap RB$ and let $x' \in (x+\varepsilon B)\cap K_i$. Then for all $s \in [0,T]$, $b \in B$, $h \in [0,\varepsilon]$ and for all j such that $-\eta'_R \leq g_i^j(x') \leq 0$

$$g_i^j(x' + h(v_{x,s}+\varepsilon b)) \leq g_i^j(x') + \nabla g_i^j(x')h(v_{x,s}+\varepsilon b) + A_{R+1}h^2|v_{x,s}+\varepsilon b|^2 \leq$$

$$\leq -h\frac{\rho_R}{2} + A_{R+1}h\varepsilon + A_{R+1}h^2|v_{x,s}+\varepsilon b|^2.$$

Therefore

$$g_i^j(x' + h(v_{x,s}+\varepsilon b)) \leq -h\frac{\rho_R}{2} + h\frac{\rho_R}{4} = -h\frac{\rho_R}{4}.$$

Consider j satisfying

$$g_i^j(x') < -\eta'_R.$$

By the assumption (A3)-i), $g_i^j(\cdot)$ is A_{R+1}-Lipschitz on $(R+1)B$, $i = 1, 2, ..., j = 1, 2, ..., m$. Therefore, for any $h \in [0,\varepsilon]$, $s \in [0,T]$, $b \in B$ and $x' \in (x+\varepsilon B)\cap K_i \cap RB$

$$g_i^j(x' + h(v_{x,s}+\varepsilon b)) \leq g_i^j(x') + A_{R+1}h|v_{x,s}+\varepsilon b| \leq -\eta'_R + A_{R+1}h(c(1+R)+1).$$

Thus

$$g_i^j(x' + h(v_{x,s}+\varepsilon b)) \leq 0.$$

Hence, for any $i \geq i_0$, $x' \in (x+\varepsilon B)\cap K_i$, $x' + [0,\varepsilon](v_{x,s}+\varepsilon B) \subset K_i$.

Set $\eta_R = \dfrac{\eta'_R}{A_{R+1}+1}$ and pick any $x \in (\partial K_i + \eta_R B)\cap K_i \cap RB$.

Then there exists j and $y \in \partial K_i$ such that $g_i^j(y) = 0$ and $|x-y| \leq \eta_R$. Therefore $0 \geq g_i^j(x) \geq -A_{R+1}\eta_R \geq -\eta'_R$.

Which ends the proof. □

Proposition 6.3.2. *Let the assumptions of Lemma 6.3.1 hold true. Then for any $\delta > 0$ there exists i_0 such that for any $i \geq i_0$*

$$K \cap RB \subset (K_i \cap (RB + \delta B)) + \delta B.$$

Proof. The proof proceeds by a contradiction argument. Assume that for some $0 < \delta < 1$ there exists a subsequence $x_{i_k} \in K\cap RB$, $k = 1, 2, ...$, such that $x_{i_k} \notin (K_{i_k}\cap(RB+\delta B))+\delta B$ and $\lim_{k\to\infty} x_{i_k} = x$, for some $x \in K \cap RB$. By the definition of K it follows $g^j(x) \leq 0$, for any j.

If $I(x) = \emptyset$, then there exists $\lambda > 0$ such that for any sufficiently large i and any j we have $g_i^j(x) < -\lambda$. Taking into consideration that $\lim_{k\to\infty} x_{i_k} = x$ and by (A3)-i) we deduce that for some $\bar{k} > 0$ and for any $k > \bar{k}$, $j = 1, ..., m$

$$g_{i_k}^j(x_{i_k}) \leq g_{i_k}^j(x) + A_{R+1}|x - x_{i_k}| < -\lambda + \frac{\lambda}{2} = -\frac{\lambda}{2}.$$

Which leads to a contradiction.

If $I(x) \neq \emptyset$, then there exists $\mu > 0$, such that for any $j \notin I(x)$ we have $g^j(x) < -\mu$ and for some $i_0 \geq 0$, for any $i > i_0$, $g_i^j(x) < -\dfrac{\mu}{2}$. Fix $s \in [0, T]$ and define $F(x) = cof(s, x, U)$. By the assumptions we can find $v_x \in F(x)$ such that $\langle \nabla g^j(x), v_x \rangle \leq -\rho_R$, for any $j \in I(x)$. Thus there exist $h \in (0, \frac{\delta}{c(|x|+1)})$ and $\theta > 0$ such that $g^j(x + hv_x) < -\theta$ for any j. As $\lim_{k\to\infty} x_{i_k} = x$ and by (A3)-i) for some $k_0 > 0$ and for any $k > k_0$

$$g^j_{i_k}(x_{i_k} + hv_x) \leq g^j_{i_k}(x + hv_x) + A_{R+1}|x - x_{i_k}| < -\frac{\theta}{2} + \frac{\theta}{4} = -\frac{\theta}{2}.$$

As $h \in (0, \frac{\delta}{c(|x|+1)})$ and $|v_x| \leq c(|x| + 1)$, thus we deduce that

$$x_{i_k} + hv_x \in K_{i_k} \cap (RB + \delta B).$$

Hence

$$x_{i_k} \in K_{i_k} \cap (RB + \delta B) + \delta B,$$

which leads to a contradiction.

□

Proposition 6.3.3. *Let (A3) holds true and $\nabla g(x) \neq 0, \forall x \in \partial K$. Then for any $R > 0$, $\delta > 0$ there exists i_0 such that for any $i \geq i_0$*

$$\overline{K_i^c \cap RB} \subset (\overline{K^c} \cap RB) + \delta B.$$

Proof. Our assumptions imply that

$$\overline{K^c} = \{x \in \mathbb{R}^n : g^j(x) \geq 0 \ for\ some\ j\}.$$

The proof proceeds by a contradiction argument. Assume for some $\delta > 0$ there exists a subsequence $x_{i_k} \in \overline{K^c_{i_k} \cap RB}$, $k = 1, 2, ...$ converging to some $x \in RB$ such that $x_{i_k} \notin (\overline{K^c} \cap RB) + \delta B$.

Taking a subsequence and keeping the same notations, we may assume that for some $j \in \{1, ..., m\}$, $g^j_{i_k}(x_{i_k}) \geq 0$, for all k.

Passing to the limit when $k \to \infty$ by (A3) we obtain that $g^j(x) \geq 0$, thus $x \in \overline{K^c} \cap RB$. Which is a contradiction, as $x_{i_k} \notin (\overline{K^c} \cap RB) + \delta B$ and $\lim_{k\to\infty} x_{i_k} = x$.

Which ends the proof.

□

Proposition 6.3.4. *Let (A3) holds true and $\nabla g(x) \neq 0, \forall x \in \partial K$. For all $x_0 \in IntK$ and $r > 0$ such that $x_0 + rB \subset K$ there exists $i(x_0)$ satisfying $x_0 + \frac{r}{2}B \subset K_i$ for all $i \geq i(x_0)$.*

Proof. We proceed by a contradiction argument. Suppose there exist $y_{i_k} \in x_0 + \frac{r}{2}B$ converging to some y such that $y_{i_k} \notin K_{i_k}$.

Then

$$y \in x_0 + \frac{r}{2}B \subset x_0 + rB \subset K.$$

Let $R > 0$ be such that $y_{i_k} \in RB$ for any $k \geq 1$. We have that for any $\delta > 0$

$$\frac{r}{2} \leq dist(y, \overline{K^c}) \leq dist(y, \overline{K^c} \cap RB) \leq dist(y, \overline{K^c} \cap RB + \delta B) + \delta.$$

From Proposition 6.3.3 we deduce that for any $\delta > 0$ there exists i_0 such that for any $i_k \geq i_0$

$$dist(y, \overline{K^c} \cap RB + \delta B) \leq dist(y, \overline{K^c_{i_k} \cap RB}).$$

Let us now estimate the right hand side of the last inequality

$$dist(y, \overline{K_{i_k}^c \cap RB}) \le dist(y_{i_k}, \overline{K_{i_k}^c \cap RB}) + |y_{i_k} - y|.$$

Hence for any $\delta > 0$ and for any $i_k \ge \delta$

$$\frac{r}{2} \le dist(y, \overline{K^c}) \le dist(y_{i_k}, \overline{K_{i_k}^c \cap RB}) + |y_{i_k} - y| + \delta.$$

Since $y_{i_k} \in \overline{K_{i_k}^c \cap RB}$ and $y_{i_k} \to y$, when $k \to \infty$, taking δ small enough leads to a contradiction with the condition $r > 0$. Which ends the proof. □

For all $(t, x) \in [0, T] \times \mathbb{R}^n$ and $i \ge 1$ define

$$G_i(t, x) \doteq \{(f_i(t, x, u), l_i(t, x, u) + r) : u \in U, r \ge 0\}.$$

Theorem 6.3.5. *Let (A3), (A4) hold true and assume that $G_i(t, x)$ is convex and closed for all $i \ge 1$, $(t, x) \in [0, T] \times \mathbb{R}^n$ and f, f_i, l, l_i satisfy (A1), (A2) with the same integrable functions $c_R(\cdot)$, absolutely continuous functions $a_R(\cdot)$ and $c > 0$. Assume that f_i converge to f, l_i converge to l and φ_i converge to φ uniformly on compacts, when $i \to \infty$ and that for some $M_R > 0$ and all $(t, x, u) \in [0, T] \times RB \times U$, $|l(t, x, u)| + |l_i(t, x, u)| \le M_R$. Then, for all $x_0 \in IntK$ and $r > 0$ such that $x_0 + rB \subset K$, $V_i \mid_{[0,T]\times B(x_0, \frac{r}{2})}$ converge uniformly to $V \mid_{[0,T]\times B(x_0, \frac{r}{2})}$, when $i \to \infty$. Furthermore for any $Q > 0$, $V_i \mid_{[0,T]\times(B(0,Q)\cap K_i)}$ are equicontinuous uniformly in i.*

Proof. We first show that $V_i \mid_{[0,T]\times(B(0,Q)\cap K_i)}$ are equicontinuous uniformly in i, for any $Q > 0$.

Fix $Q > 0$. Let us now prove that there exist increasing, continous functions $\omega' : [0, +\infty) \to [0, +\infty)$ and $\omega'' : [0, +\infty) \to [0, +\infty)$ with $\omega'(0) = 0$ and $\omega''(0) = 0$ such that for any $t_1, t_0 \in [0, T]$ and $y_1, y_2 \in B(0, Q) \cap K_i$.

$$\begin{cases} |V_i(t_0, y_1) - V_i(t_0, y_2)| \le \omega'(|y_1 - y_2|) \\ |V_i(t_1, y_1) - V_i(t_0, y_1)| \le \omega''(|t_1 - t_0|). \end{cases} \tag{6.3.9}$$

By our assumptions $\varphi_i(\cdot)$ are equicontinous on compact subsets of $\mathbb{R}^n$. Thus for any compact set $\Omega \subset \mathbb{R}^n$ there exists an increasing, continuous function $\omega_\Omega : [0, +\infty) \to [0, +\infty)$ such that $\omega_\Omega(0) = 0$ and $|\varphi_i(x) - \varphi_i(y)| \le \omega_\Omega(|x - y|)$, for all $x, y \in \Omega$. Let us prove that there exists a modulus of continuity $\omega'(\cdot)$ such that for all i and for any $y_1, y_2 \in B(0, Q) \cap K_i$ and $t_0 \in [0, T]$

$$|V_i(t_0, y_1) - V_i(t_0, y_2)| \le \omega'(|y_1 - y_2|).$$

Let $i(y_1)$ be as in Proposition 6.3.4. It is well known that, taking into account Lemma 6.3.1, under assumptions of Theorem 6.3.5 there exist $(y_i(\cdot), u_i(\cdot)) \in S^i_{[t_0,T]}(y_1)$, for any $i \ge i_0$ such that

$$V_i(t_0, y_1) = \int_{t_0}^{T} l_i(s, y_i(s), u_i(s))\, ds + \varphi_i(y_i(T)).$$

Then $y_i(\cdot)$ is a trajectory of the following system

$$\begin{cases} \dot{y}_i(s) = f_i(s, y_i(s), u_i(s)) \\ \dot{\bar{z}}_i(s) = l_i(s, y_i(s), u_i(s)) \\ y_i(t_0) = y_1 \\ \bar{z}_i(t_0) = 0, \end{cases}$$

satisfying $y_i(s) \in K_i$, for all $s \in [t_0, T]$.

Now consider the solution $(x_i(\cdot), z_i(\cdot))$ of

$$\begin{cases} \dot{x}_i(s) = f_i(s, x_i(s), u_i(s)) \\ \dot{z}_i(s) = l_i(s, x_i(s), u_i(s)) \\ x_i(t_0) = y_2 \\ z_i(t_0) = 0. \end{cases}$$

Let $R > 0$ be such that for every trajectory-control pair (x, u) of the control system (6.1.2) with B replaced by U and f by f_i satisfying $x(t_0) \in B(0, Q)$ for some $t_0 \in [0, T]$ we have $x_i(T) \in B(0, R)$. We would like to underline that $B(0, R)$ depends on Q. As $Q > 0$ is fixed, for the simplicity we will omit the subindex $B(0, R)$ for $\omega_{B(0,R)}(\cdot)$. By Lemma 6.3.1 and Theorem 6.2.1 applied to

$$F(t, x, z) = \{(f_i(t, x, u), l_i(t, x, u)), u \in U\},$$

there exists $C > 0$ independent from i such that for all $i \geq 1$ we can find absolutely continuous $(\tilde{x}_i(\cdot), \tilde{z}_i(\cdot))$ such that $(\dot{\tilde{x}}_i(t), \dot{\tilde{z}}_i(t)) \in F(t, \tilde{x}_i(t), \tilde{z}_i(t))$ a.e. $\tilde{x}_i(t_0) = y_2, \tilde{z}_i(t_0) = 0$ satisfying state constraints $(\tilde{x}_i(t), \tilde{z}_i(t)) \in K_i \times \mathbb{R}$, for all $t \in [t_0, T]$ such that

$$\|\tilde{z}_i - z_i\|_\infty + \|\tilde{x}_i - x_i\|_\infty \leq C \max_{s \in [t_0, T]} dist(x_i(s), K_i).$$

By the Gronwall inequality for any $s \in [t_0, T]$ and for a constant $E > 0$ we have that

$$\|x_i(s) - y_i(s)\|_\infty \leq E|y_2 - y_1|.$$

Using Gronwall's inequality and the fact that $y_i(s) \in K_i$, for all $s \in [t_0, T]$ we deduce that

$$\max_{s \in [t_0, T]} dist(x_i(s), K_i) \leq \max_{s \in [t_0, T]} |x_i(s) - y_i(s)| \leq E|y_2 - y_1|.$$

By the Filippov theorem [9, Theorem 8.2.10] for some measurable $\tilde{u}_i(\cdot) : [t_0, T] \to U$ we have

$$\begin{cases} \dot{\tilde{x}}_i(s) = f_i(s, \tilde{x}_i(s), \tilde{u}_i(s)) \ a.e. \ in \ [t_0, T] \\ \dot{\tilde{z}}_i(s) = l_i(s, \tilde{x}_i(s), \tilde{u}_i(s)) \ a.e. \ in \ [t_0, T]. \end{cases}$$

We already know that

$$|\varphi_i(\tilde{x}_i(T)) - \varphi_i(x_i(T))| \leq \omega(|\tilde{x}_i(T) - x_i(T)|)$$

and also

$$\left|\int_{t_0}^T l_i(s, \tilde{x}_i(s), \tilde{u}_i(s))\, \mathrm{d}s - \int_{t_0}^T l_i(s, x_i(s), u_i(s))\, \mathrm{d}s\right| \leq \|\tilde{z}_i - z_i\|_\infty \leq CE|y_2 - y_1|.$$

By the definition of $V_i(t_0, \cdot)$

$$V_i(t_0, y_2) - V_i(t_0, y_1) \leq \int_{t_0}^T l_i(s, \tilde{x}_i(s), \tilde{u}_i(s))\, \mathrm{d}s + \varphi_i(\tilde{x}_i(T)) - \int_{t_0}^T l_i(s, y_i(s), u_i(s))\, \mathrm{d}s -$$

$$\varphi_i(y_i(T)) = \int_{t_0}^T (l_i(s, \tilde{x}_i(s), \tilde{u}_i(s)) - l_i(s, y_i(s), u_i(s)))\, \mathrm{d}s + (\varphi_i(\tilde{x}_i(T)) - \varphi_i(y_i(T))).$$

Hence

$$V_i(t_0,y_2) - V_i(t_0,y_1) \leq \int_{t_0}^{T} (l_i(s,\tilde{x}_i(s),\tilde{u}_i(s)) - l_i(s,x_i(s),u_i(s)))\,\mathrm{d}s +$$

$$+\int_{t_0}^{T} (l_i(s,x_i(s),u_i(s)) - l_i(s,y_i(s),u_i(s)))\,\mathrm{d}s + \varphi_i(\tilde{x}_i(T)) - \varphi_i(x_i(T)) +$$

$$+\varphi_i(x_i(T)) - \varphi_i(y_i(T)).$$

From $(A1)$ we deduce that

$$V_i(t_0,y_2) - V_i(t_0,y_1) \leq CE|y_2-y_1| + \int_{t_0}^{T} c_R(s)|x_i(s)-y_i(s)|\,\mathrm{d}s + \omega(|\tilde{x}_i(T)-x_i(T)|) +$$

$$+\omega(|x_i(T)-y_i(T)|).$$

Thus for some $\bar{M}>0$ and $\bar{c}>0$ independent from i and for all $y_1,y_2 \in B(0,Q)\cap K$

$$V_i(t_0,y_2) - V_i(t_0,y_1) \leq \bar{M}|y_2-y_1| + \omega(\bar{c}|y_2-y_1|).$$

For all $s \geq 0$ define $\omega'_Q(s) \doteq \bar{M}s + \omega(\bar{c}s)$.

Thus

$$V_i(t_0,y_2) - V_i(t_0,y_1) \leq \omega'_Q(|y_2-y_1|).$$

Interchanging the roles of y_1,y_2 we deduce that

$$|V_i(t_0,y_2) - V_i(t_0,y_1)| \leq \omega'_Q(|y_2-y_1|). \tag{6.3.10}$$

Which ends the proof of the first inequality in (6.3.9).

Now let us show that $V_i\mid_{[0,T]\times(B(0,Q)\cap K_i)}$ are equicontinuous with respect to the time variable too. Consider $y_0 \in B(0,Q)\cap K_i$, $0 \leq t_0 < t_1 \leq T$, $(x_i(\cdot),u_i(\cdot)) \in S^i_{[t_0,T]}(y_0)$ such that

$$V_i(t_0,y_0) = \int_{t_0}^{T} l_i(s,x_i(s),u_i(s))\,\mathrm{d}s + \varphi_i(x_i(T)).$$

We have that for some $\bar{c}>0$ independent from i and y_0

$$|x_i(t_1)-y_0| = |x_i(t_1)-x_i(t_0)| \leq \bar{c}|t_1-t_0|.$$

Since $y_0 \in B(0,Q)\cap K$, we deduce that $x_i(t_1) \in B(0,Q+\bar{c}T)$.

By the dynamic programming principle

$$V_i(t_0,y_0) = \int_{t_0}^{t_1} l_i(s,x_i(s),u_i(s))\,\mathrm{d}s + V_i(t_1,x_i(t_1)).$$

Hence

$$V_i(t_1,x_i(t_1)) - V_i(t_0,y_0) = -\int_{t_0}^{t_1} l_i(s,x_i(s),u_i(s))\,\mathrm{d}s. \tag{6.3.11}$$

Therefore

$$|V_i(t_0,y_0) - V_i(t_1,y_0)| \leq |V_i(t_0,y_0) - V_i(t_1,x_i(t_1))| + |V_i(t_1,x_i(t_1)) - V_i(t_1,y_0)| \leq$$

$$\leq \int_{t_0}^{t_1} |l_i(s,x_i(s),u_i(s))|\,\mathrm{d}s + |V_i(t_1,x_i(t_1)) - V_i(t_1,y_0)|.$$

We have that l_i are equi-bounded on compacts. Therefore for some $c > 0$ independent from i by (6.3.11) we deduce that

$$\int_{t_0}^{t_1} |l_i(s, x_i(s), u_i(s))| \, ds \le c|t_1 - t_0|.$$

Let $\bar{Q}$ be such that every trajectory $x(\cdot)$ of (6.1.2) with f replaced by f_i with $x([0,T]) \cap B(0,Q) \neq \emptyset$ satisfies $x([0,T]) \subset B(0,\bar{Q})$.

According to (6.3.10) it follows that

$$|V_i(t_1, x_i(t_1)) - V_i(t_1, y_0)| \le \omega'_{\bar{Q}}(|x_i(t_1) - y_0|).$$

Thus

$$|V_i(t_0, y_0) - V_i(t_1, y_0)| \le c|t_1 - t_0| + \omega'_{\bar{Q}}(|x_i(t_1) - y_0|).$$

Therefore

$$|V_i(t_0, y_0) - V_i(t_1, y_0)| \le c|t_1 - t_0| + \omega'_{\bar{Q}}(\bar{c}|t_1 - t_0|).$$

Set $\omega''(s) \doteq cs + \omega'_{\bar{Q}}(\bar{c}s)$ for all $s \ge 0$. Hence we have proved that for all $0 \le t_0 < t_1 < T$

$$|V_i(t_0, y_0) - V_i(t_1, y_0)| \le \omega''(|t_1 - t_0|).$$

Therefore, $V_i \mid_{[0,T]\times(B(0,Q)\cap K_i)}$ are equicontinuous uniformly in i, for any $Q > 0$.

Fix $(t_0, x_0) \in [0,T] \times IntK$. Let $r > 0$ be such that $x_0 + rB \subset K$ and $y_0 \in x_0 + \frac{r}{2}B$. We claim that

$$\lim_{i\to\infty} V_i(t_0, y_0) = V(t_0, y_0).$$

First we will show that

$$V(t_0, y_0) \le \liminf_{i\to\infty} V_i(t_0, y_0).$$

Let $i(y_0)$ be as in Proposition 6.3.4. It is well known that, taking into account Lemma 6.3.1, under assumptions of Theorem 6.3.5 there exist $(x_i, u_i) \in S^i_{[t_0,T]}(y_0)$, for any $i \ge i_0$ such that

$$V_i(t_0, y_0) = \varphi_i(x_i(T)) + \int_{t_0}^{T} l_i(s, x_i(s), u_i(s)) \, ds.$$

Consider a subsequence V_{i_j} such that

$$\liminf_{i\to\infty} V_i(t_0, y_0) = \lim_{j\to\infty} V_{i_j}(t_0, y_0).$$

By (A2) we may assume that i_j are such that x_{i_j} converge uniformly on $[t_0, T]$ to an absolutely continuous function $x \in W^{1,1}([t_0,T];\mathbb{R}^n)$, $\dot{x}_{i_j}(\cdot)$ converge weakly in L^1 to $\dot{x}$ and

$$\xi_j(\cdot) \doteq l_{i_j}(\cdot, x_{i_j}(\cdot), u_{i_j}(\cdot))$$

converges weakly in L^1 to some $\psi(\cdot)$. Then

$$\int_{t_0}^{T} l_{i_j}(s, x_{i_j}(s), u_{i_j}(s)) \, ds \to \int_{t_0}^{T} \psi(s) \, ds.$$

By our assumptions for any $R > 0$ and for every $\varepsilon > 0$, there exists $i_0 \ge 1$ such that for any $i \ge i_0$, $t \in [0,T]$, $x \in RB$, $u \in U$ and $\varepsilon > 0$ we have

$$|l_i(t,x,u) - l(t,x,u)| \le \varepsilon,$$

and

$$|f_i(t,x,u) - f(t,x,u)| \le \varepsilon.$$

Fix $\varepsilon > 0$ and denote

$$G_\varepsilon(t,x) \doteq G(t,x) + \varepsilon B.$$

Then $G_\varepsilon(t,x)$ is closed and convex.

As $x_{i_j}(\cdot) \to x(\cdot)$ uniformly on $[t_0,T]$, there exists $R>0$ such that $\|x_{i_j}(\cdot)\|_\infty \le R$ for all j. Using Lipschitzianity assumptions (A1) we deduce that for all sufficiently large j and all $t \in [t_0,T]$

$$(f_{i_j}(t, x_{i_j}(t), u_{i_j}(t)), l_{i_j}(t, x_{i_j}(t), u_{i_j}(t))) \in G_\varepsilon(t, x(t)) + 2c_R(t)|x_{i_j}(t) - x(t)|B.$$

For all $t \in [t_0,T]$ the sets

$$Q_\varepsilon(t) \doteq G_\varepsilon(t,x(t)) + 2c_R(t)\varepsilon B$$

are convex and closed.

Thus the set $\{v(\cdot) \in L^1([t_0,T];\mathbb{R}^n) : v(t) \in Q_\varepsilon(t), \forall\, t \in [t_0,T]\}$ is convex and closed in L^1.

By the Mazur theorem (applied in L^1) it follows $(\dot{x}(s), \psi(s)) \in Q_\varepsilon(s)$ a.e. since $\varepsilon > 0$ is arbitrary, we get $(\dot{x}(s),\psi(s)) \in G(s,x(s))$ a.e. in $[t_0,T]$. By the measurable selection theorem there exist a measurable selection $u(s) \in U$ and $\lambda(s) \ge 0$

$$\begin{cases} \dot{x}(s) = f(s,x(s),u(s)) \\ \psi(s) = l(s,x(s),u(s)) + \lambda(s). \end{cases}$$

Since $\psi(\cdot) \in L^1$ and l is bounded on compacts, $\lambda(\cdot)$ is integrable. Notice that, as $(x_i,u_i) \in S^i_{[t_0,T]}(y_0)$, for any $i \ge i_0$ and $x_{i_j}(\cdot) \to x(\cdot)$ uniformly on $[t_0,T]$, hence $x(t) \in K$, for any $t \in [t_0,T]$. We have that

$$\lim_{j\to\infty} V_{i_j}(t_0,y_0) = \varphi(x(T)) + \int_{t_0}^T l(s,x(s),u(s))\,\mathrm{d}s + \int_{t_0}^T \lambda(s)\,\mathrm{d}s \ge V(t_0,y_0).$$

We show next that

$$V(t_0,y_0) \ge \limsup_{i\to\infty} V_i(t_0,y_0).$$

Let $(\bar{x}(\cdot),\bar{u}(\cdot)) \in S_{[t_0,T]}(y_0)$ be such that

$$V(t_0,y_0) = \varphi(\bar{x}(T)) + \int_{t_0}^T l(s,\bar{x}(s),\bar{u}(s))\,\mathrm{d}s,$$

and for almost all $s \in [t_0,T]$

$$\begin{cases} \dot{\bar{x}}(s) = f(s,\bar{x}(s),\bar{u}(s)) \\ \dot{z}(s) = l(s,\bar{x}(s),\bar{u}(s)) \\ \bar{x}(t_0) = y_0, \\ z(t_0) = 0, \\ \bar{x}(s) \in K,\ \ s \in [t_0,T]. \end{cases}$$

Then $(\bar{x}(s), z(s)) \in K \times \mathbb{R}$, for all $s \in [t_0,T]$.

Consider the solutions $x_i(\cdot)$ of

$$\begin{cases} \dot{x}_i(s) = f_i(s, x_i(s), \bar{u}(s)) \\ \dot{z}_i(s) = l_i(s, x_i(s), \bar{u}(s)) \\ x_i(t_0) = y_0 \\ z_i(t_0) = 0, \end{cases}$$

for $i = 1, 2, ...$

Observe that for any $\varepsilon > 0$ there exists $\bar{i}_0 > 0$, such that for any $i > \bar{i}_0$ we have $|x_i - \bar{x}|_\infty \leq \varepsilon$ and $|z_i - z|_\infty \leq \varepsilon$. Let R be such that $R > |\bar{x}|_\infty$. For any $\delta > 0$ and for any sufficiently large i, by triangular inequality it follows that

$$dist((x_i(s), z_i(s)), (K_i \times \mathbb{R})) = dist(x_i(s), K_i) \leq dist(x_i(s), K_i \cap (RB + \delta B)) \leq$$

$$\leq dist(x_i(s), (K_i \cap (RB + \delta B)) + \delta B) + \delta.$$

Fix $\delta > 0$. From Proposition 6.3.2 it follows that there exists $i_0 > 0$, such that for any $i > i_0$

$$dist(x_i(s), (K_i \cap (RB + \delta B)) + \delta B) \leq dist(x_i(s), K \cap RB).$$

Hence for all sufficiently large i

$$dist((x_i(s), z_i(s)), (K_i \times \mathbb{R})) \leq dist(x_i(s), K \cap RB) + \delta.$$

Consequently, for any $\delta > 0$ there exists i_0, such that for any $i > i_0$

$$dist((x_i(s), z_i(s)), (K_i \times \mathbb{R})) \leq 2\delta.$$

By Lemma 6.3.1 and Theorem 6.2.1 applied to

$$F(t, x, z) = \{(f_i(t, x, u), l_i(t, x, u)), u \in U\}$$

there exists $C > 0$ independent from i such that for all sufficiently large i we can find absolutely continuous $(\tilde{x}_i(\cdot), \tilde{z}_i(\cdot))$ such that $(\dot{\tilde{x}}_i(t), \dot{\tilde{z}}_i(t)) \in F(t, \tilde{x}_i(t), \tilde{z}_i(t))$ a.e. in $[t_0, T]$, $\tilde{x}_i(t_0) = y_0$, $\tilde{z}_i(t_0) = 0$ satisfying state constraints $(\tilde{x}_i(t), \tilde{z}_i(t)) \in K_i \times \mathbb{R}$, such that

$$\|\tilde{z}_i - z_i\|_\infty + \|\tilde{x}_i - x_i\|_\infty \leq C \max_{s \in [t_0, T]} dist(x_i(s), K_i).$$

We have that for any $\delta > 0$

$$\max_{s \in [t_0, T]} dist(x_i(s), K_i) \leq \max_{s \in [t_0, T]} dist(x_i(s), K_i \cap (R + \delta)B).$$

From Proposition 6.3.2 we deduce for any $\delta > 0$ and all sufficiently large i that

$$\max_{s \in [t_0, T]} dist(x_i(s), K_i) \leq \max_{s \in [t_0, T]} dist(x_i(s), K \cap RB) + \delta \leq \|x_i - \bar{x}\|_\infty + \delta \leq \varepsilon + \delta.$$

Thus taking $\delta = \varepsilon$ we deduce that

$$\|\tilde{z}_i - z_i\|_\infty + \|\tilde{x}_i - x_i\|_\infty \leq 2C\varepsilon.$$

Consider measurable $\tilde{u}_i(\cdot) : [t_0, T] \to U$ such that

$$\begin{cases} \dot{\tilde{x}}_i(s) = f_i(s, \tilde{x}_i(s), \tilde{u}_i(s)) \ a.e. \ in \ [t_0, T] \\ \dot{\tilde{z}}_i(s) = l_i(s, \tilde{x}_i(s), \tilde{u}_i(s)) \ a.e. \ in \ [t_0, T]. \end{cases}$$

For any $\varepsilon > 0$ there exists $\tilde{i}_0 > 0$, such that for any $i > \tilde{i}_0$ we have

$$|\varphi_i(\tilde{x}_i(T)) - \varphi(\bar{x}(T))| \leq \varepsilon$$

and

$$|\int_{t_0}^{T} l_i(s, \tilde{x}_i(s), \tilde{u}_i(s))\,\mathrm{d}s - \int_{t_0}^{T} l(s, \bar{x}(s), \bar{u}(s))\,\mathrm{d}s| \leq \varepsilon.$$

Hence we obtain

$$V(t_0, y_0) = \int_{t_0}^{T} l(s, \bar{x}(s), \bar{u}(s))\,\mathrm{d}s + \varphi(\bar{x}(T)) \geq$$

$$\geq \int_{t_0}^{T} l_i(s, \tilde{x}_i(s), \tilde{u}_i(s))\,\mathrm{d}s + \varphi_i(\tilde{x}_i(T)) - 2\varepsilon \geq V_i(t_0, y_0) - 2\varepsilon.$$

Thus

$$V(t_0, y_0) \geq \limsup_{i \to \infty} V_i(t_0, y_0) - 2\varepsilon.$$

The above being valid for any $\varepsilon > 0$, therefore we get

$$V(t_0, y_0) \geq \limsup_{i \to \infty} V_i(t_0, y_0).$$

Having that for any $Q > 0$, V_i are equicontinuous uniformly in i on $[0,T] \times (B(0,Q) \cap K_i)$ and converging pointwise to V on $[0,T] \times B(x_0, \frac{r}{2})$ we deduce that the convergence is uniform.

The proof is complete. □

Corollary 6.3.6. *Let the assumptions of Theorem 6.3.5 hold true. Then*

$$Lim_{i \to \infty} graph V_i = graph V,$$

where the limit is taken in the Kuratowski sense.

Proof. We will first prove that

$$graph V \subset Liminf_{i \to \infty} graph V_i.$$

Case 1. Let $(t,x) \in [0,T] \times Int K$. We will show that

$$((t,x), V(t,x)) \in Liminf_{i \to \infty} graph V_i.$$

Take any (relatively) open neighborhood Ω of $((t,x), V(t,x))$ in $[0,T] \times \mathbb{R}^n \times \mathbb{R}$. It is not restrictive to assume that $\Omega = W_0 \times U_0$, where W_0 is an open neighborhood of (t,x) and U_0 is an open neighborhood of $V(t,x)$.

By Theorem 6.3.5 for all $x \in Int K$ and $r > 0$ such that $x + rB \subset K$ we have $V_i(\cdot,\cdot) \to V(\cdot,\cdot)$ uniformly on $[0,T] \times B(x, \frac{r}{2})$, when $i \to \infty$, thus there exists an open neighborhood W_1 of (t,x) and there exists $i_0 \geq 1$ such that for any $(s,y) \in W_1$ and any $i \geq i_0$, $V_i(s,y) \in U_0$. Therefore

$$((W_1 \cap W_0) \times U_0) \cap graph V_i \neq \emptyset,$$

for any $i \geq i_0$. We deduce that $\Omega \cap graph V_i \neq \emptyset$, for any $i \geq i_0$.

Hence for any $(t,x) \in [0,T] \times Int K$

$$((t,x), V(t,x)) \in Liminf_{i \to \infty} graph V_i.$$

Case 2. Let $(t,x) \in [0,T] \times \partial K$. Take any open neighborhood Ω of $((t,x),V(t,x))$. It is not restrictive to assume that $\Omega = W_0 \times U_0$, where W_0 is an open neighborhood of (t,x) and U_0 is an open neighborhood of $V(t,x)$. There exists $x_1 \in IntK$, such that $(t,x_1) \in W_0$ and $V(t,x_1) \in U_0$ (by continuity of $V(t,\cdot)$ on K). Thus, we can choose W_1 an open neighborhood of (t,x_1) and U_1 an open neighborhood of $V(t,x_1)$, such that $W_1 \times U_1 \subseteq W_0 \times U_0$. Consider $\Omega_1 = W_1 \times U_1$, then $\Omega_1 \subseteq \Omega$. As $x_1 \in IntK$, by the result of Case 1 we have that there exists $i_0 \geq 1$ such that $\Omega_1 \cap graphV_i \neq \emptyset$, for any $i \geq i_0$. Therefore $\Omega \cap graphV_i \neq \emptyset$, for any $i \geq i_0$. Hence for any $(t,x) \in [0,T] \times \partial K$

$$((t,x),V(t,x)) \in Liminf_{i\to\infty} graphV_i,$$

Combining the results of Case 1 and Case 2 we deduce that

$$graphV \subset Liminf_{i\to\infty} graphV_i. \tag{6.3.12}$$

In order to complete the proof let us now prove that

$$graphV \supset Limsup_{i\to\infty} graphV_i.$$

Take any $\omega \in Limsup_{i\to\infty} graphV_i$, thus for any open neighborhood $Q \ni \omega$ we have $Q \cap graphV_i \neq \emptyset$, for infinitely many i. Thus $B(\omega, \frac{1}{k}) \cap graphV_i \neq \emptyset$, for infinitely many i, where $B(\omega, \frac{1}{k})$ is the ball of center ω and with the radius $\frac{1}{k}$, for any $k > 0$. Hence there exist $v_{i_k} \in B(\omega, \frac{1}{k}) \cap graphV_{i_k}$, such that $v_{i_k} = ((t_{i_k}, x_{i_k}), V_{i_k}(t_{i_k}, x_{i_k}))$, for some $(t_{i_k}, x_{i_k}) \in [0,T] \times K$.

Therefore

$$|((t_{i_k}, x_{i_k}), V_{i_k}(t_{i_k}, x_{i_k})) - \omega| < \frac{1}{k}.$$

Let $v \in \mathbb{R}$ be such that $\omega = ((t,x),v)$, for some $(t,x) \in [0,T] \times \mathbb{R}^n$. Hence for any $k > 0$ we have that

$$\begin{cases} |x_{i_k} - x| < \dfrac{1}{k} \\ |t_{i_k} - t| < \dfrac{1}{k} \\ |V_{i_k}(t_{i_k}, x_{i_k}) - v| < \dfrac{1}{k}. \end{cases} \tag{6.3.13}$$

By (6.3.12) it follows that there exist $(\bar{t}_k, \bar{x}_k) \in [0,T] \times K_{i_k}$, $(\bar{t}_k, \bar{x}_k) \to (t,x)$, when $k \to \infty$, such that

$$V_{i_k}(\bar{t}_k, \bar{x}_k) \to V(t,x). \tag{6.3.14}$$

From (6.3.13) we have that when $k \to \infty$, then $(t_{i_k}, x_{i_k}) \to (t,x)$.

By triangular inequality

$$\begin{cases} |t_{i_k} - \bar{t}_k| \leq |t_{i_k} - t| + |t - \bar{t}_k| \\ |x_{i_k} - \bar{x}_k| \leq |x_{i_k} - x| + |x - \bar{x}_k| \end{cases} \tag{6.3.15}$$

and

$$|V_{i_k}(t_{i_k}, x_{i_k}) - V(t,x)| \leq |V_{i_k}(t_{i_k}, x_{i_k}) - V_{i_k}(\bar{t}_k, \bar{x}_k)| + |V_{i_k}(\bar{t}_k, \bar{x}_k) - V(t,x)|. \tag{6.3.16}$$

Since (by Theorem 6.3.5) $V_{i_k} \mid_{[0,T]\times K_{i_{i_k}}}$ are equicontinuous (in the sense of Definition 6.2.2), then by (6.3.16), (6.3.15) and (6.3.14) we deduce that

$$\lim_{k\to\infty} V_{i_k}(t_{i_k}, x_{i_k}) = V(t,x).$$

Hence and by (6.3.13) we obtain that $v = V(t,x)$.
Therefore $((t,x), V(t,x)) = \omega$, thus $\omega \in graphV$.
Thus

$$Lim_{i\to\infty} graphV_i = Liminf_{i\to\infty} graphV_i = Limsup_{i\to\infty} graphV_i = graphV.$$

Which ends the proof. □

6.4 HJB equations and the Bolza optimal control problem

Let K be a closed nonempty subset of $\mathbb{R}^n$. Consider the Hamilton-Jacobi equation

$$(HJB) \begin{cases} -V_t(t,x) + H(t,x,-V_x(t,x)) = 0, \ (t,x) \in [0,T] \times K \\ V(T,x) = \varphi(x), \end{cases} \tag{6.4.1}$$

with the Hamiltonian $[0,T] \times \mathbb{R}^n \times \mathbb{R}^n \ni (t,x,p) \to H(t,x,p)$.

Definition 6.4.1. *For a map $H : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, H^* denotes the conjugate of H with respect to the third variable, i.e. for all $(t,x,v) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$*

$$H^*(t,x,v) \doteq \sup_{p\in\mathbb{R}^n} \{\langle v,p\rangle - H(t,x,p)\} \in \mathbb{R} \cup \{+\infty\}.$$

Assumptions.
(H1) $H(t,x,\cdot)$ is convex for any $(t,x) \in [0,T] \times \mathbb{R}^n$.
(H2) For any $R > 0$ there exists an integrable $c_R : [0,T] \to \mathbb{R}_+$ such that for all $x, y \in RB$, $t \in [0,T]$ and $p \in \mathbb{R}^n$

$$|H(t,x,p) - H(t,y,p)| \le c_R(t)(1 + |p|)|x - y|.$$

(H3) There exists $c > 0$ such that

$$|H(t,x,p) - H(t,x,q)| \le c(1 + |x|)|p - q|$$

for all $(t,x) \in [0,T] \times \mathbb{R}^n$ and $p, q \in \mathbb{R}^n$.
(H4) $H^*(t,x,\cdot)$ is bounded on its domain for all $(t,x) \in [0,T] \times \mathbb{R}^n$.
(H5) For every $R > 0$ there exists $M_R > 0$ such that for all $(t,x) \in [0,T] \times RB$ and $v \in dom(H^*(t,x,\cdot))$ we have

$$H^*(t,x,v) = \max_{p\in B(0,M_R)} (\langle v,p\rangle - H(t,x,p)).$$

(H6) For every $R > 0$ there exists an absolutely continuous $a_R : [0,T] \to \mathbb{R}$ such that for all $x \in RB,\ p \in \mathbb{R}^n$ and $t,\ s \in [0,T]$

$$|H(t,x,p) - H(s,x,p)| \le (1 + |p|)|a_R(t) - a_R(s)|.$$

Definition 6.4.2. *A continuous function $W : [0,T] \times K \to \mathbb{R}$ is called a viscosity solution of* (6.4.1) *if $W(T,\cdot) = \varphi(\cdot)$ and*
i) for all $(s,x) \in (0,T) \times K$ and all $(p_s, p_x) \in \partial_- W(s,x)$

$$-p_s + H(s,x,-p_x) \ge 0.$$

ii) for all $(s,x) \in (0,T) \times IntK$ and all $(p_s,p_x) \in \partial_+ W(s,x)$

$$-p_s + H(s,x,-p_x) \le 0.$$

We have shown in [40] that if $(H1)-(H6)$ hold true, then there exist $f:[0,T]\times\mathbb{R}^n\times B\to\mathbb{R}^n$ and $l:[0,T]\times\mathbb{R}^n\times B\to\mathbb{R}$ satisfying $(A1)-(A2)$ with $U=B$ and such that $f(t,x,B)=domH^*(t,x,\cdot)$,

$$H(t,x,p)=\max_{u\in B}(\langle p,f(t,x,u)\rangle-l(t,x,u)).$$

Moreover

$$G(t,x)=\{(f(t,x,u),l(t,x,u)+r):u\in B,r\geq 0\}$$

is convex and closed.

Let V be the value function defined in Section 6.2 for f,l and U as above.

Proposition 6.4.3. *For all* $(s,x)\in[0,T)\times K$ *and all* $(p_s,p_x)\in\partial_-V(s,x)$

$$-p_s+H(s,x,-p_x)\geq 0.$$

Proof. Fix $(t_0,x_0)\in[0,T)\times K$ and let $(\bar{x}(\cdot),\bar{u}(\cdot))$ be optimal for (P) at (t_0,x_0), therefore

$$V(t,\bar{x}(t))=V(t_0,\bar{x}(t_0))-\int_{t_0}^{t}l(s,\bar{x}(s),\bar{u}(s))\,\mathrm{d}s.$$

Take $t\doteq t_0+h$ with $h>0$ small enough. Hence

$$\frac{V(t_0+h,\bar{x}(t_0+h))-V(t_0,\bar{x}(t_0))}{h}=-\frac{1}{h}\int_{t_0}^{t_0+h}l(s,\bar{x}(s),\bar{u}(s))\,\mathrm{d}s. \tag{6.4.2}$$

We shall deduce that for some $(v,\gamma)\in G(t_0,x_0)$

$$D_\uparrow V(t_0,\bar{x}(t_0))(1,v)\leq-\gamma.$$

For this aim consider $h_i\to 0+$, when $i\to\infty$ and $v\in\mathbb{R}^n$, $\gamma\in\mathbb{R}$, such that

$$\begin{cases}\dfrac{\bar{x}(t_0+h_i)-\bar{x}(t_0)}{h_i}\to v\\ \dfrac{\int_{t_0}^{t_0+h_i}l(s,\bar{x}(s),\bar{u}(s))\,\mathrm{d}s}{h_i}\to\gamma.\end{cases} \tag{6.4.3}$$

We deduce from the continuity of f, l and $(A2)$ that for any $\varepsilon>0$ there exists $h_0>0$ such that for any $s\in[t_0,t_0+h_0]$

$$(f(s,\bar{x}(s),\bar{u}(s)),l(s,\bar{x}(s),\bar{u}(s)))\subset(f(t_0,\bar{x}(t_0),\bar{u}(s)),l(t_0,\bar{x}(t_0),\bar{u}(s)))+\varepsilon B\subset$$

$$\subset G(t_0,x_0)+\varepsilon B.$$

Hence $(v,\gamma)\in G(t_0,x_0)$.

Thus from Definition 6.2.3, (6.4.2), (6.4.3) we deduce that

$$D_\uparrow V(t_0,x_0)(1,v)\leq-\gamma. \tag{6.4.4}$$

By definition of $G(\cdot,\cdot)$, there exists u_0 and $r_0\geq 0$ such that

$$\begin{cases}v=f(t_0,x_0,u_0),\\ \gamma=l(t_0,x_0,u_0)+r_0.\end{cases} \tag{6.4.5}$$

From (6.4.4) and (6.4.5) we obtain that

$$D_{\uparrow}V(t_0,x_0)(1,f(t_0,x_0,u_0)) \leq -l(t_0,x_0,u_0) - r_0 \leq -l(t_0,x_0,u_0).$$

For any $(p_s,p_x) \in \partial_- V(t_0,x_0)$ using the Lemma 6.2.7 we obtain that

$$p_s \cdot 1 + \langle p_x, f(t_0,x_0,u_0)\rangle \leq D_{\uparrow}V(t_0,x_0)(1,f(t_0,x_0,u_0)) \leq -l(t_0,x_0,u_0).$$

Thus

$$-p_s + \langle -p_x, f(t_0,x_0,u_0)\rangle - l(t_0,x_0,u_0) \geq 0,$$

and we obtain

$$-p_s + \sup_{u\in B}(\langle -p_x, f(t_0,x_0,u)\rangle - l(t_0,x_0,u)) \geq 0.$$

Hence for any $(p_s,p_x) \in \partial_- V(t_0,x_0)$

$$-p_s + H(t_0,x_0,-p_x) \geq 0.$$

Since $(t_0,x_0) \in [0,T) \times K$ is arbitrary, we end the proof. □

Proposition 6.4.4. *For all $(s,x) \in [0,T) \times IntK$ and all $(p_s,p_x) \in \partial_+ V(s,x)$*

$$-p_s + H(s,x,-p_x) \leq 0.$$

Proof. Fix $u_0 \in B$ and consider the solution $x(\cdot)$ of

$$\begin{cases} \dot{x}(s) = f(s,x(s),u_0) \\ x(t_0) = x_0. \end{cases}$$

Then

$$V(t_0+h, x(t_0+h)) \geq V(t_0,x_0) - \int_{t_0}^{t_0+h} l(s,x(s),u_0)\,\mathrm{d}s.$$

We have that for $h \to 0+$

$$\frac{x(t_0+h)-x_0}{h} \to f(t_0,x_0,u_0)$$

and

$$\frac{1}{h}\int_{t_0}^{t_0+h} l(s,x(s),u_0)\,\mathrm{d}s \to l(t_0,x_0,u_0).$$

By Lemma 6.2.7 for any $(p_s,p_x) \in \partial_+ V(t_0,x_0)$ we have that

$$\langle (p_s,p_x),(1,f(t_0,x_0,u_0))\rangle \geq D_{\downarrow}V(t_0,x_0)(1,f(t_0,x_0,u_0)) \geq -l(t_0,x_0,u_0).$$

Hence, we have obtained that for any $(p_s,p_x) \in \partial_+ V(t_0,x_0)$ and $u_0 \in B$

$$-p_s + \langle -p_x, f(t_0,x_0,u_0)\rangle - l(t_0,x_0,u_0) \leq 0,$$

and therefore for any $(p_s,p_x) \in \partial_+ V(t_0,x_0)$

$$-p_s + H(t_0,x_0,-p_x) \leq 0.$$

Since $(t_0,x_0) \in [0,T) \times K$ is arbitrary, we end the proof. □

Theorem 6.4.5. *If assumptions $(H1)-(H6)$ hold true, then the value function of the Bolza optimal control problem is a viscosity solution of the Hamilton-Jacobi equation,* (6.4.1).

Proof. By Theorem 6.3.5 the value function is continuous on $[0,T] \times K$. According to Definition 6.4.2 and Proposition 6.4.3 the value function is a viscosity supersolution of Hamilton-Jacobi equation and by Proposition 6.4.4 the value function is a viscosity subsolution of Hamilton-Jacobi equation, thus it is a viscosity solution. Which ends the proof. □

6.5 Uniqueness of solution. Continuous dependence on data

Theorem 6.5.1. *Let assumptions (H1)-(H6) hold true and*

$(A4)_H$. *For any $R > 0$ there exist $\rho_R > 0$ such that for every $x \in K \cap RB$ with $I(x) \neq \emptyset$ and every $t \in [0, T]$*

$$\inf_{v \in dom(H^*(t,x,\cdot))} \max_{j \in I(x)} \langle \nabla g^j(x), v \rangle \leq -\rho_R.$$

Then there exists the unique viscosity solution of the Hamilton-Jacobi equation (6.4.1) *on* $[0, T] \times K$.

Proof. We have shown in [40] that if $(H1) - (H6)$ hold true for H, then there exist $f : [0, T] \times \mathbb{R}^n \times B \to \mathbb{R}^n$ and $l : [0, T] \times \mathbb{R}^n \times B \to \mathbb{R}$ satisfying $(A1) - (A2)$ with $U = B$ and such that

$$H(t, x, p) = \max_{u \in B}(\langle p, f(t, x, u)\rangle - l(t, x, u)).$$

Moreover

$$G(t, x) = \{(f(t, x, u), l(t, x, u) + r) : u \in B, r \geq 0\}$$

is convex and closed.

We consider the Bolza optimal control problem (6.3.1) with $U = B$ and the associated value function. By Theorem 6.4.5 we know that the value function is a viscosity solution of the Hamilton-Jacobi-Bellman equation.

Let W be a viscosity solution of (6.4.1). We have to show that $W = V$ on $[0, T] \times K$. We will proceed in 2 steps.

Step 1. We show first that for any $(t_0, x_0) \in [0, T] \times K$, it holds true

$$W(t_0, x_0) \geq V(t_0, x_0).$$

Since W is a viscosity solution, by Definition 6.4.2 we have

$$\begin{cases} \forall (t, x) \in (0, T) \times K,\ \forall (p_t, p_x) \in \partial_- W(t, x), \\ -p_t + \sup_{u \in B}(\langle -p_x, f(t, x, u)\rangle - l(t, x, u)) \geq 0. \end{cases} \tag{6.5.1}$$

If for some $(t, x) \in (0, T) \times K$ and $z \geq W(t, x)$

$$(p_t, p_x, q) \in N_{epi(W)}(t, x, z),$$

then $(p_t, p_x, q) \in N_{epi(W)}(t, x, W(t, x))$. By [31, Lemma 4.2] there exist $(t_i, x_i) \in (0, T) \times K$, such that $(t_i, x_i) \to (t, x)$, when $i \to \infty$ and

$$(p_t^i, p_x^i, q_i) \in N_{epi(W)}(t_i, x_i, W(t_i, x_i)), \tag{6.5.2}$$

where $q_i < 0$ and such that

$$(p_t^i, p_x^i, q_i) \to (p_t, p_x, q),\ \textit{when}\ i \to \infty.$$

Therefore, as $q_i < 0$, we deduce from (6.5.2) that

$$\Big(\frac{p_t^i}{|q_i|}, \frac{p_x^i}{|q_i|}, -1\Big) \in N_{epi(W)}(t_i, x_i, W(t_i, x_i)).$$

Hence, by [31, Proposition 4.1], we obtain that

$$\Big(\frac{p_t^i}{|q_i|}, \frac{p_x^i}{|q_i|}\Big) \in \partial_- W(t_i, x_i). \tag{6.5.3}$$

From (6.5.1) and (6.5.3) we deduce that the following inequality holds true

$$-\frac{p_t^i}{|q_i|} + \sup_{u\in B}\Big(\langle -\frac{p_x^i}{|q_i|}, f(t_i, x_i, u)\rangle - l(t_i, x_i, u)\Big) \geq 0$$

or equivalently

$$-p_t^i + \sup_{u\in B}\Big(\langle -p_x^i, f(t_i, x_i, u)\rangle - |q_i| l(t_i, x_i, u)\Big) \geq 0.$$

Passing to the limit when $i \to \infty$, by continuity of f and l we obtain that

$$-p_t + \sup_{u\in B}\Big(\langle -p_x, f(t, x, u)\rangle - |q| l(t, x, u)\Big) \geq 0.$$

Therefore

$$p_t + \inf_{u\in B}\Big(\langle p_x, f(t, x, u)\rangle + |q| l(t, x, u)\Big) \leq 0. \tag{6.5.4}$$

Consider a solution x of

$$\begin{cases} \dot{x}(s) = f(s, x(s), u(s)),\ s \in [0, T],\ u(s) \in B \\ x(0) = x_0 \in RB \cap K. \end{cases} \tag{6.5.5}$$

From (6.5.5) and (A2) together with the Gronwall Lemma, it follows that there exists $c > 0$ such that

$$\sup_{t\in[t_0,T]} |x(t)| \leq e^{cT}|x_0| < 2e^{cT}R \doteq \hat{R}.$$

Therefore any solution starting at $x_0 \in B(0, R)$ and defined on $[t_0, T]$ stays in $\mathring{B}(0, \hat{R})$. For any $(t, x, u) \in [0, T] \times B(0, 2\hat{R}) \times B$ denote by

$$M \doteq \max_{(t,x,u)\in[0,T]\times B(0,2\hat{R})\times B} |l(t, x, u)|,$$

as l is continuous and $[0, T] \times B(0, 2\hat{R}) \times B$ is a compact set, thus $M > 0$ is a constant, such that for any $(t, x, u) \in [0, T] \times B(0, 2\hat{R}) \times B$ we have

$$|l(t, x, u)| \leq M.$$

Define a set-valued map $F^- : [0, T] \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ by

$$F^-(t, x, v) \doteq \{(1, f(t, x, u), -l(t, x, u) - r) \mid u \in B, r \in [0, M - l(t, x, u)]\},$$

where M is as above. Notice that F^- has convex compact images.

Let us prove that

$$F^-(t, x, v) \cap \bar{co}T_{epi(W)}(t, x, z) \neq \emptyset, \tag{6.5.6}$$

for any $(t, x) \in (0, T) \times (K \cap B(0, e^{cT}R))$, $z \geq W(t, x)$.

We proceed by a contradiction argument. Indeed, if (6.5.6) is not satisfied for some $(t, x, v) \in (0, T) \times (K \cap B(0, e^{cT}R)) \times B$, then by the separation theorem there exists

$$0 \neq (p_t, p_x, q) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R},$$

such that

$$\inf_{(\alpha,\beta,\gamma)\in F^-(t,x,v)} \langle(\alpha,\beta,\gamma),(p_t,p_x,q)\rangle > \sup_{w\in \bar{co}T_{epi(W)}(t,x,W(t,x))} \langle w,(p_t,p_x,q)\rangle \geq 0. \tag{6.5.7}$$

Notice that if we assume that the right hand side of (6.5.7) is not equal to 0, then it is equal to $+\infty$ since the supremum is taken over a cone, leading to a contradiction because the left hand side of (6.5.7) is bounded. Thus, we deduce that

$$\sup_{w\in \bar{co}T_{epi(W)}(t,x,W(t,x))} \langle w,(p_t,p_x,q)\rangle = 0. \tag{6.5.8}$$

Hence, from (6.5.7) and (6.5.8) we obtain that for all $r \in [0, M - l(t,x,u)]$

$$p_t + \langle p_x, f(t,x,u)\rangle + q(-l(t,x,u) - r) > 0. \tag{6.5.9}$$

From (6.5.8) it follows that

$$(p_t,p_x,q) \in N_{epi(W)}(t,x,W(t,x)). \tag{6.5.10}$$

Therefore, from (6.5.10) we deduce that $q \leq 0$, thus by (6.5.9) we obtain

$$p_t + \langle p_x, f(t,x,u)\rangle + |q|(l(t,x,u) + r) > 0.$$

Let us take $r = 0$, hence

$$p_t + \langle p_x, f(t,x,u)\rangle + |q|l(t,x,u) > 0.$$

This leads to a contradiction with (6.5.4).

Hence (6.5.6) holds true.

Consider the control system

$$(CS1)\begin{cases} \dot t(s) = 1 \\ \dot x(s) = f(t_0+s, x(s), u(s)),\ \ u(s)\in B \\ \dot z(s) = -l(t_0+s, x(s), u(s)) - r(s),\ \ r(s)\in[0, M - l(s,x(s),u(s)] \end{cases}$$

We have that W is continuous, thus $epi(W)$ is closed. On the other hand, F^- is continuous and has convex compact images, thus by [7, Theorem 3.2.4] and [7, Local Viability Theorem 3.3.4] we deduce that for any $(t_0,x_0)\in(0,T)\times K$ there exists a solution $(t(\cdot),x(\cdot),z(\cdot))$ of (CS1) on $[0,T-t_0]$ such that $t(0)=t_0, x(0)=x_0, z(0)=W(t_0,x_0)$ and

$$(t(s),x(s),z(s)) \in epi(W),$$

for any $s\in[0,T-t_0)$.

Therefore we have for any $s\in[0,T-t_0)$ that

$$z(s) \geq W(t(s),x(s)). \tag{6.5.11}$$

By continuity it holds true also for $s = T - t_0$.

Take $s = T - t_0$, thus we obtain from (6.5.11) that

$$z(T-t_0) \geq W(t(T-t_0), x(T-t_0)). \tag{6.5.12}$$

We set

$$y(t_0+s) \doteq x(s),$$

hence

$$x(T-t_0) = y(T)$$

and

$$W(t(T-t_0), x(T-t_0)) = W(T, y(T)).$$

From (6.5.12) we deduce that for any $(t_0, x_0) \in (0,T) \times K$

$$W(t_0, x_0) - \int_0^{T-t_0} l(t_0+\tau, y(t_0+\tau), u(\tau))\,\mathrm{d}\tau \geq \varphi(y(T)).$$

We set $\hat{u}(t_0+s) \doteq u(s)$. Therefore

$$W(t_0, x_0) - \int_{t_0}^{T} l(s, y(s), \hat{u}(s))\,\mathrm{d}s \geq \varphi(y(T)).$$

Hence, for any $(t_0, x_0) \in (0,T) \times K$

$$W(t_0, x_0) \geq \varphi(y(T)) + \int_{t_0}^{T} l(s, y(s), \hat{u}(s))\,\mathrm{d}s \geq V(t_0, x_0).$$

Using that W and V are continuous we end the proof of Step 1.

Step 2. We show next that for any $(t_0, x_0) \in [0,T] \times K$, it holds true

$$W(t_0, x_0) \leq V(t_0, x_0).$$

Since W is a viscosity solution, by Definition 6.4.2 we have that

$$\begin{cases} \forall (t,x) \in (0,T) \times IntK, \ \forall (p_t, p_x) \in \partial_+ W(t,x), \\ -p_t + \sup\limits_{u \in B}(\langle -p_x, f(t,x,u)\rangle - l(t,x,u)) \leq 0. \end{cases} \tag{6.5.13}$$

Let us prove the following claim.

Claim 1. For any $(t,x) \in (0,T) \times IntK$ and $u \in B$

$$(1, f(t,x,u), -l(t,x,u)) \in \bar{co}T_{hyp(W)}(t,x,z),$$

for any $z \leq W(t,x)$.

We proceed by a contradiction argument. Suppose there exists $u_0 \in B$, such that for $z = W(t,x)$ we have that

$$(1, f(t,x,u_0), -l(t,x,u_0)) \notin \bar{co}T_{hyp(W)}(t,x,W(t,x)).$$

By the separation Theorem we deduce that there exists

$$0 \neq (p_t, p_x, q) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}.$$

such that

$$\sup_{w \in \bar{co}T_{hyp(W)}(t,x,W(t,x))} \langle (p_t, p_x, q), w\rangle < \langle (p_t, p_x, q), (1, f(t,x,u_0), -l(t,x,u_0))\rangle. \tag{6.5.14}$$

Notice that the left hand side of (6.5.14) can not be positive, because the maximum over the cone on the left hand side is bounded.

Therefore

$$\sup_{w \in \bar{co} T_{hyp(W)}(t,x,W(t,x))} \langle w, (p_t, p_x, q) \rangle = 0. \tag{6.5.15}$$

From (6.5.14) we also deduce that $q \geq 0$.

Therefore

$$p_t + \langle p_x, f(t,x,u_0) \rangle - ql(t,x,u_0) > 0. \tag{6.5.16}$$

By (6.5.15)

$$(p_t, p_x, q) \in N_{hyp(W)}(t, x, W(t,x)).$$

By [31, Lemma 4.2] (substituting epigraph by hypograph) there exist $(t_i, x_i) \in (0,T) \times K$, such that $(t_i, x_i) \to (t, x)$, when $i \to \infty$ and

$$(p_t^i, p_x^i, q_i) \in N_{hyp(W)}(t_i, x_i, W(t_i, x_i)), \tag{6.5.17}$$

where $q_i > 0$ and such that

$$(p_t^i, p_x^i, q_i) \to (p_t, p_x, q), \ \ when \ i \to \infty.$$

Therefore, as $q_i > 0$, we deduce from (6.5.17) that

$$\Big(\frac{p_t^i}{q_i}, \frac{p_x^i}{q_i}, 1\Big) \in N_{hyp(W)}(t_i, x_i, W(t_i, x_i)).$$

Hence, by [31, page 267], we obtain that

$$\Big(-\frac{p_t^i}{q_i}, -\frac{p_x^i}{q_i}\Big) \in \partial_+ W(t_i, x_i). \tag{6.5.18}$$

From (6.5.13) and (6.5.18) we deduce that

$$\frac{p_t^i}{q_i} + \langle \frac{p_x^i}{q_i}, f(t_i, x_i, u_0) \rangle - l(t_i, x_i, u_0) \leq 0,$$

or equivalently

$$p_t^i + \langle p_x^i, f(t_i, x_i, u_0) \rangle - q_i l(t_i, x_i, u_0) \leq 0.$$

Passing to the limit when $i \to \infty$, by continuity of f and l we obtain that

$$p_t + \langle p_x, f(t,x,u_0) \rangle - ql(t,x,u_0) \leq 0.$$

This is a contradiction with (6.5.16).

Which ends the proof of Claim 1.

Claim 2. For any $(t,x) \in (0,T) \times IntK$ and $z \leq W(t,x)$, any $u \in B$

$$(1, f(t,x,u), -l(t,x,u)) \in T_{hyp(W)}(t,x,z).$$

The proof of Claim 2 follows from Lemma 6.2.6 and the Claim 1.

Consider the control system

$$(CS2) \begin{cases} \dot{t}(s) = 1 \\ \dot{x}(s) = f(t_0 + s, x(s), u(s)), \ \ u(s) \in B \\ \dot{z}(s) = -l(t_0 + s, x(s), u(s)). \end{cases}$$

From the proof of [31, Theorem 3.3] (substituting *epigraph* by *hypograph*) we deduce that the set

$$\Psi = hyp(W) \cap (0,T) \times IntK \times \mathbb{R},$$

is locally invariant by the system (CS2), i.e. for any solution $(t(\cdot), x(\cdot), z(\cdot))$ of (CS2) with $t(0) = t_0 \in (0,T)$, $x(0) = x_0 \in IntK$, $z(0) = W(t_0, x_0)$, satisfying $x(s) \in IntK$, $s \in [0,\delta]$, for some $\delta > 0$ we have

$$(t(s), x(s), z(s)) \in hyp(W).$$

Therefore, we deduce that

$$z(s) \le W(t(s), x(s)).$$

Hence

$$W(t_0, x_0) - \int_{t_0}^{t_0+\delta} l(t_0 + s, x(s), u(s))\,\mathrm{d}s \le W(t_0 + \delta, x(t_0 + \delta)).$$

Thus, if a solution $(x,u)(\cdot)$ of (CS2) satisfies $x(s) \in IntK$ on $[t_1, t_2]$, then

$$W(t_1, x(t_1)) \le W(t_2, x(t_2)) + \int_{t_1}^{t_2} l(s, x(s), u(s))\,\mathrm{d}s.$$

Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be optimal for (P) at $(t_0, x_0) \in (0,T) \times IntK$. By Theorem 6.2.1 applied to (CS2) and $\mathcal{K} = hyp(W)$ there exist controls u_ε such that $x_\varepsilon(\cdot)$ corresponding to u_ε converge uniformly to $\bar{x}(\cdot)$, when $\varepsilon \to 0$ and $z_\varepsilon(\cdot)$ defined on $[t_0, T]$ by

$$z_\varepsilon(t) \doteq W(t_0, x_0) - \int_{t_0}^{t} l(s, x_\varepsilon(s), u_\varepsilon(s))\,\mathrm{d}s$$

converge uniformly to $z(\cdot)$ given by

$$z(t) \doteq W(t_0, x_0) - \int_{t_0}^{t} l(s, \bar{x}(s), \bar{u}(s))\,\mathrm{d}s$$

and for all $t \in (t_0, T]$

$$(t, x_\varepsilon(t), z_\varepsilon(t)) \in Int(hyp(W)).$$

Hence $x_\varepsilon(t) \in IntK$ on $(t_0, T]$.

Therefore, we deduce that for any $t \in (t_0, T]$

$$z_\varepsilon(t) \le W(t, x_\varepsilon(t)).$$

Hence for all small $\tau > 0$

$$W(t_0 + \tau, x_\varepsilon(t_0 + \tau)) - \int_{t_0+\tau}^{T} l(s, x_\varepsilon(s), u_\varepsilon(s))\,\mathrm{d}s \le W(T, x_\varepsilon(T)) = \varphi(x_\varepsilon(T)).$$

Taking the limit when $\tau \to 0+$ we get

$$W(t_0, x_\varepsilon(t_0)) - \int_{t_0}^{T} l(s, x_\varepsilon(s), u_\varepsilon(s))\,\mathrm{d}s \le W(T, x_\varepsilon(T)) = \varphi(x_\varepsilon(T)).$$

Passing to the limit when $\varepsilon \to 0+$ we deduce that

$$W(t_0, x_0) \le \varphi(\bar{x}(T)) + \int_{t_0}^{T} l(s, \bar{x}(s), \bar{u}(s))\,\mathrm{d}s.$$

We obtain that for any $(t_0, x_0) \in (0, T) \times IntK$

$$W(t_0, x_0) \leq V(t_0, x_0).$$

Since W and V are continuous, we end the proof of step 2.

From Step 1 and Step 2 we deduce that the value function of the Bolza problem is the unique viscosity solution of the Hamilton-Jacobi equation on $[0, T] \times K$ (in the class of continuous functions).

Which ends the proof of Theorem 6.5.1.

□

Theorem 6.5.2. *For every $i \geq 1$ let K_i and K be closed nonempty subsets of $\mathbb{R}^n$ defined by* (6.3.7), (6.3.8) *respectively and (A3) holds true. Consider continuous $H_i : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfying the assumptions $(H1) - (H6)$ with the same integrable functions $c_R(\cdot)$, absolutely continuous functions $a_R(\cdot)$ and $c > 0$, $M_R > 0$. Assume that for some $H : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $H_i \to H$ uniformly on compacts, when $i \to \infty$ and that assumption $(A4)_H$ holds true. Consider viscosity solutions W_i to Hamilton-Jacobi equation* (6.4.1) *with H replaced by H_i and K replaced by K_i. Let $x_0 \in IntK$, $r > 0$ such that $B(x_0, r) \subset K$. Then the restrictions of W_i to $[0, T] \times B(x_0, \frac{r}{2})$ converge uniformly to the restriction to $[0, T] \times B(x_0, \frac{r}{2})$ of the unique solution W of* (6.4.1).

Proof. Clearly H satisfyies $(H1)-(H6)$ with the same $c_R(\cdot)$, $a_R(\cdot)$, c, M_R. We have shown in [40] that if $(H1) - (H6)$ hold true for H and H_i, then there exists f, f_i, l, l_i satisfying $(A1) - (A2)$ such that

$$H(t, x, p) = \max_{u \in B}(\langle p, f(t, x, u)\rangle - l(t, x, u))$$

and

$$H_i(t, x, p) = \max_{u \in B}(\langle p, f_i(t, x, u)\rangle - l_i(t, x, u)).$$

Moreover

$$G_i(t, x) = \{(f_i(t, x, u), l_i(t, x, u) + r) : u \in B, r \geq 0\}$$

is convex and closed. Let $x_0 \in IntK$ and $r > 0$ be such that $B(x_0, r) \subset K$. By Theorems 6.4.5 and 6.5.1 the value function of the Bolza problem (with f_i, l_i) is the unique viscosity solution of the Hamilton-Jacobi equation on $[0, T] \times K_i$. As $H_i \to H$ uniformly on compacts, when $i \to \infty$, thus by [40, Theorem 4.1] we have that f_i converge to f and l_i converge to l uniformly on compacts, when $i \to \infty$. Proposition 6.3.4 and Theorem 6.3.5 finish the proof. □

Corollary 6.5.3. *Let the assumptions of Theorem 6.5.2 hold true. Then*

$$Lim_{i \to \infty} epiW_i = epiW,$$

where W is the unique viscosity solution of (6.4.1).

Proof. The proof follows by Corollary 6.3.6 and from the fact that $W_i = V_i$ is a bounded family of equicontinuous functions.

□

6.6 Acknowledgments

This work was co-funded by the European Union under the 7th Framework Programme "FP7-PEOPLE-2010-ITN", grant agreement number 264735-SADCO.